



Regular article

A novel approach towards the robustness of centrality measures in networks

Jens Dörpinghaus^{a,b,*}, Vera Weil^{a,c}, Robert Rockenfeller^d, Meetkumar Pravinbhai Mangroliya^d^a Federal Institute for Vocational Education and Training (BIBB), Bonn, Germany^b Department of Computer Science, University of Koblenz, Germany^c Department of Mathematics and Computer Science, University of Cologne, Germany^d Mathematical Institute, University of Koblenz, Germany

ARTICLE INFO

Keywords:

Social network analysis
 Knowledge graphs
 Digital humanities
 Centrality measures
 Error measures

ABSTRACT

Centrality measures were first introduced in the Social Network Analysis (SNA), which is widely used in the humanities and social sciences. However, recently they are receiving attention in network science in general, e.g., for knowledge graphs and artificial intelligence. Assuming that the network is not error-free or contains multiple layers, we can identify challenges for their quality. For example, historical and narrative texts in ancient languages are usually challenging for natural language processing methods and artificial intelligence technologies developed for modern languages due to their complexity and missing models. In addition, due to quiet sources, nodes and edges in the network may be missing, which might influence the results of SNA. Other aspects may also have an effect on data, as many networks include additional layers, e.g. spatial information or archaeological artifacts. This extends to knowledge graphs. In this paper, we will summarize, compare and evaluate existing and novel methods to analyze the robustness of networks. We introduce a method with different removal strategies to analyze how additional or missing layers, nodes, or edges in a random network influence centrality measures. We can show that the robustness of social networks regardless of the error measures heavily relies on network structures, which brings up several new challenges for future research. In general, we may assume networks to be rather robust against small errors and few missing data if they follow a scale-free distribution. The results of this paper are not limited to social networks but can be applied in all fields working with centrality measures.

1. Introduction

Knowledge graphs and networks in general have been shown to play an important role in recent knowledge mining and discovery, for example in the fields of digital humanities, life sciences or bioinformatics. They also include single-purpose networks, like social networks, but mostly they also contain additional information and data, see for example Berhan, Adams, McKether, and Kumar (2019), Rollinger (2014), Suárez, Díaz-Puente, and Bettoni (2021). This paper underlines the fact that the research on social networks naturally extends to knowledge graphs and graphs in general. It summarizes, compares, and evaluates methods for analyzing network robustness and introduces a novel approach that uses different removal strategies to assess the impact of additional or missing layers, nodes, or edges on centrality measures. We show that network robustness, regardless of error measures, is highly dependent on network structure, which presents new challenges for

future research. Our results suggest that networks following a scale-free distribution are generally robust to small errors and missing data, with implications beyond social networks for any field using centrality measures. Therefore, our findings are not contingent on the specific types of networks (e.g., social, biological, or transportation) or the centrality measures examined (degree, betweenness, closeness, etc.). This conclusion is further supported by the illustrative examples analyzed in our study.

1.1. Background

A knowledge graph can be seen as a multi-layer graph comprising different data layers, for example social data, spatial data, etc.¹ In addition, scientists study network patterns and structures, for example, paths, communities or other patterns within the data structure,

* Corresponding author at: Federal Institute for Vocational Education and Training (BIBB), Bonn, Germany.

E-mail addresses: jens.doerpinghaus@bibb.de (J. Dörpinghaus), weil@cs.uni-koeln.de (V. Weil), rockenfeller@uni-koblenz.de (R. Rockenfeller).

¹ According to Kivelä, Arenas, Barthelemy, Gleeson, Moreno, and Porter (2014) a multilayer network is a quadruplet $M = (V_M, E_M, V, \mathbb{L})$ where $G_M = (V_M, E_M)$ is a labeled graph, $\mathbb{L} = \{L_a\}_{a=1}^d$ sets of elementary layers and $V_M = V \times L_1 \times \dots \times L_d$. However, also other definitions were proposed. In Dörpinghaus, Klante, Christian, Meigen, and Düing (2022a) the equivalence of the different approaches was shown.

see for example Dörpinghaus and Stefan (2019). Very few studies have investigated the questions of how missing data and multiple layers within graphs impact methods and algorithms developed for single-purpose networks, see Rossetti, Citraro, and Milli (2021). Social Network Analysis (SNA) also utilizes knowledge graphs and is an emerging topic, not only in the social sciences but also in historical and religious research for the last years, see Reitmayer and Marx (2010), Rollinger (2020). Here it is often called Historical or Religious Network Analysis (HNA, RNA). Understanding social interactions and networks in how they influence society are important issues. In the last few years there has been a growing interest in using social networks in historical sciences (Jackson, 2010; Ruegg & Lee, 2020; Zhitomirsky-Geffet & Prebor, 2019). Quite recently, considerable attention has been paid to social networks in religious studies and especially in theology. Although SNA and HNA are emerging topics, most works show that the described data and source problems are the greatest hurdles. A lack of data also prevents automatic data generation, see Leidwanger et al. (2014). Therefore, instead of speaking of historical network research, researchers suggest the term “using network approaches”. This seems to be reasonable, since Collar, among others, suggests a deliberately simplified definition of “network thinking as a new methodology for understanding the processes of change and the spread of innovation in the past” (Collar, 2013, p. 6). Thus, as a second problem, the method criticism is widely discussed. Rollinger, for example, deals extensively with method criticism and identifies the variable range of results beside the source problem as a reason for quality problems (Rollinger, 2014). In summary, textual, literal or historical sources need to be interpreted because they do not contain direct statements about the quality of the relationships or their temporal duration. While network analysis can show additional aspects, working with the original sources is crucial (Ganter, 2015). Although some preliminary work has been carried out in this field (de Valeriola, 2021; Dörpinghaus, Weil, Düing, & Sommer, 2022b), there are still several open questions regarding the robustness of social networks for missing and additional data. For example, no work has been carried out on multi-layer networks and while different measures were proposed, e.g., Kendall’s tau, no work compared these approaches and their sensitivity. We will also discuss how these concepts depend on the context (e.g., social or knowledge networks).

1.2. Objectives

In this paper, we study the robustness of two key centrality measures – degree centrality and betweenness centrality – in the context of social network analysis. Our focus is on understanding how these centrality measures are affected by the addition or removal of nodes and edges, which can be interpreted either as missing or additional data or as representing another layer of data in a multi-layer network. Our main research question is: How robust are degree and betweenness centrality measures when applied to social networks, particularly under scenarios involving incomplete or evolving data? We focus on these centrality measures due to their significant roles in network analysis, acknowledging that other network measures are also important but are beyond the scope of this study. To quantitatively evaluate the impact of data changes on degree and betweenness centrality measures, we introduce a novel measure and employ well-known non-parametric statistical tests such as the Kolmogorov–Smirnov (K–S) test and the Earth Mover’s Distance (EMD). These tests are specifically chosen for their ability to assess changes in the degree distribution of the network before and after data perturbations without assuming normality. This approach provides valuable insights into the statistical significance of the observed structural changes. Our experimental environment tests the robustness of degree and betweenness centrality on random graphs inspired by social network analysis, including small-world and scale-free networks. While our research is motivated by SNA, the findings are

broadly applicable to understanding the robustness of these centrality measures in various types of networks.

This novel approach is generic and can be applied to any networks. However, as will be demonstrated in this paper, different networks, such as scale-free or small-world networks, exhibit distinct robustness characteristics. Consequently, there is no universal solution for analyzing the robustness of any given network.

This paper is divided into five sections. This first section describes the foundations, the state of the art and related work. We will in particular introduce knowledge graphs and centrality measures. The second section gives a brief overview of materials and methods. In the third section, we present the experimental setting and the methods used for this evaluation. The fourth section is dedicated to experimental results and the evaluation. Our conclusions are drawn in the final section.

2. Foundations, review and related work

In this section, we will present some preliminaries describing the foundations of SNA and the relation between social networks and knowledge graphs. Afterward, we will introduce centrality measures as a core topic of SNA and discuss error measures comparing a network with a modified network.

2.1. Preliminaries

In general, the definition of social networks seems to be so trivial that apparently most scholars begin with examples of friendship or exchange networks, see Dörpinghaus (2021), Jackson (2010). Here, the situation seems similar to the concept of knowledge graphs which are closely related to the concept of social networks, see Dörpinghaus, Binnewitt, Winnige, Hein, and Krüger (2023), Dörpinghaus and Jacobs (2019), Dörpinghaus et al. (2022a). Many authors tried to define knowledge graphs, but still a formal definition is missing, see Fensel et al. (2020). In Ehrlinger and Wöß (2016) the authors compared several definitions, but the only formal definition was limited to RDF graphs which does not cover labeled property graphs. However, by using formal structures like extra meta information and weighted edges within a social network, we are implicitly using the model of a labeled property graph, see Rodriguez and Neubauer (2010, 2012). Thus, we follow a very generic graph-theoretical definition as proposed in Dörpinghaus et al. (2022a) and shown as equivalent to social networks:

Definition 1 (Knowledge Graph). A Knowledge Graph is a graph $G = (V, E)$ with vertices (nodes) $v \in V$ and edges (relations) $e \in E$. Both edges and vertices are part of previously well-defined, not necessarily disjoint categories, $V \subseteq C_1 \cup C_2 \cup \dots \cup C_n$ and $E \subseteq R_1 \cup R_2 \cup \dots \cup R_m$, which are elaborated in the following.

The network G may either be directed or undirected. This has an impact on the methods which can be used for analyzing the network and on the overall representation of the data. Usually, connections are multiplex, which means a relation, i.e. edge, does not necessarily belong to one distinct category: A friend might, for example, also be a co-worker. In addition, if person $\mathfrak{A}$ is a friend of $\mathfrak{B}$, it is most likely that person $\mathfrak{B}$ is a friend of $\mathfrak{A}$. However, occasionally, special cases need to be represented and it is necessary to use directed graphs. Stegbauer and Häußling point out that a categorization of ties is also highly relevant to the transitivity of ties in triangular form: Here, for three individuals $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, it holds that the stronger the friendship relationship between individuals $\mathfrak{A}$ and $\mathfrak{B}$ and individuals $\mathfrak{A}$ and $\mathfrak{C}$, the more likely that $\mathfrak{B}$ and $\mathfrak{C}$ know or will know each other (Avenarius, 2010). Easley calls this ‘Triadic Closure’ (Easley & Kleinberg, 2010).

The sets C_i contain the objects found in a social network, such as actors, organizations, and locations. For example, we may identify two sets, namely persons and locations, $V = \{C_{actors}, C_{groups}\}$, and enrich

Table 1
Selected publications covering the issue of the robustness of centrality measures.

Networks	Centrality Measures	Evaluation	
(Directed) Graphs	BC, CC, DC, EC	Bounds	Bof, Baggio, and Zampieri (2016), Segarra and Ribeiro (2015)
Real Historical/Social Networks	BC, CC, DC, EC, PR	Categories and sampling	Borgatti et al. (2006), Žnidaršič, Ferligoj, and Doreian (2018), de Valeriola (2021), Kardos et al. (2020)
Random Networks	Eleven centrality measures	Sampling	Costenbader and Valente (2003)
Multilayer Networks	DC	Random graphs	Tsugawa and Ohsaki (2015)
Random and Real World Networks	DC, BC	Random graphs & bounds	Dörpinghaus et al. (2022b)
12 empirical networks	BC, CC, DC, EC, PR	Kendall's tau	Martin and Niemeyer (2019)
12 empirical networks	6 Centrality Measures	CONCOR	Smith and Moody (2013)
12 empirical networks	6 Centrality Measures	Bias	Smith et al. (2017, 2022)
Preferential-attachment Networks	Random graphs	Network robustness	Xu and Frank (2021)
			Klau and Weiskircher (2005)
			Hasheminezhad and Brandes (2023)

All publications focus on different networks, e.g. generic graphs, real or random networks (first column), centrality measures (second column) and different evaluation processes (third column); some publications offer mathematical bounds.

them with other entities, locations and references. In turn, the sets R_i may represent different kinds of relations, for example, what kind of relation actors have, or which evidence for a relation between an actor and a location exists. We may consider positive and negative relations ($E = \{R_{pos}, E_{neg}\}$). It is also possible to model strong and weak ties, which were introduced in the work of Granovetter (1973).

2.2. Centrality measures

In this section, we consider a graph $G = (V, E)$ as a network representation of one or multiple given social networks or knowledge graph, e.g., textual or historical sources and we do not know the quality of this network. Usually in the context of social networks questions like “How many friends does actor X have?” or “How many groups does actor Y belong to?” are considered. The mathematical formulation of these questions would be “What is the degree of node X ?” and “How many communities C_i can be found so that $Y \subset C_i$?”. We will limit our analysis to measures which are most widely used, and discuss several properties to evaluate nodes and their position in the networks. These properties can be used to calculate statistical parameters, so-called *centrality measures*, cf. Freeman (1978) and Carrington, Scott, and Wasserman (2005). They answer the question “Which nodes in this network are particularly significant or important?”. For mathematical details we refer to Appendix A.

Not much work has been carried out to evaluate the robustness of centrality measures against errors in the network, see Table 1 for selected publications. A critical review has been presented in Landherr, Friedl, and Heidemann (2010). Some researchers focused on sampling the network (Costenbader & Valente, 2003), others defined categories of perturbation (Borgatti, Carley, & Krackhardt, 2006; Kardos, London, & Vinkó, 2020) or focused more on the social perspective of these measures (Düring, 2016; Zemljič & Hlebec, 2005). The study of de Valeriola (2021) addresses several transformations on original graphs to track their impact on centrality measures. However, it is also unclear, what stability means in terms of centrality measures. In Segarra and Ribeiro (2015), the authors assumed the existence of a factor K_G which relies on the graph structure of G to prove some bounds for directed graphs. Other researchers focused on the influence of network size (Martin & Niemeyer, 2020), robustness in general (Hasheminezhad & Brandes, 2023; Klau & Weiskircher, 2005) or the concept of biases (Smith, Moody, & Morgan, 2017; Smith, Morgan, & Moody, 2022; Xu & Frank, 2021). The only generic approaches proposed so far are Kendall's tau (Martin & Niemeyer, 2019) or bound based on the total or local disorder of ordered nodes (Dörpinghaus et al., 2022b). A marginal topic is the usage of AI approaches like deep learning (Ma et al., 2021). An evaluation and comparison of generic mathematical definition and error measures, e.g., upper and lower bounds, based on the graph topology, are currently missing.

Yet another issue that has not been covered in research is the existence of multiple layers on centrality measures, see Dörpinghaus

et al. (2022b). Interestingly, the idea of multilayer networks was first introduced in the social sciences (Craven & Wellman, 1973) and received attention in other domains not before the 2000s (Kivelä et al., 2014). Here, it is necessary to evaluate how graph structures and in particular measures change when additional information are stored in extra layers. This is a crucial point, in particular due to the fact that most social networks are covering multiple layers, e.g., categories.

In summary, we could identify an evaluation of generic mathematical definitions and error measures as gap in research. While Definition 1 presents a most generic definition of knowledge graphs and social networks as graphs, it is nonetheless unclear which applications of SNA meet this definition. In addition, we could identify several approaches towards error measures or error evaluation. However, none of them refers to a clear graph-theoretical definition of social networks and it is hard to generalize them. While Martin and Niemeyer proposed Kendall's tau as a solution, it was shown that this approach does not generalize and does not work well for all graphs (Martin & Niemeyer, 2020) and our research will support these results.

This issue is, however, directly linked to our second research question on the robustness of analyses, e.g., centrality measures, to analyze social networks, in particular for scenarios with additional or missing data. We will now continue with the materials and methods used for our novel approach.

3. Methods

First, we will discuss the networks to carry out the error analysis, in particular real-world networks and random graphs which are related to social networks and knowledge graphs. Second, we will introduce our evaluation approach which is not related to error categories, but to a generic random subset of a graph which identifies either ‘missing’ or ‘extra’ knowledge. This can be seen as a generalization of sampling methods, and the focus on generic random graphs as a generalization of the analysis of particular real world networks.

3.1. Real world graphs

We will describe results on three different real-world graphs. The first graph E_1 is a network of actors in Luke-Acts (Dörpinghaus, 2022; Dörpinghaus & Stenschke, 2021). It thus represents both a narrative and a historical network.

The second graph, E_2 , shows how members of the US Congress interacted on Twitter between February 9, 2022, and June 9, 2022 (Fink et al., 2023a; Fink, Omodt, Zinnecker, & Sprint, 2023b). At first, the graph was directed and weighted, but for our purposes, we only considered one direction and disregarded the weights. Therefore, we ended up with an undirected graph that retained 10,222 of the initial 13,289 edges.

The third graph, E_3 , depicts a network of individuals who trade Bitcoin on the trading platform Bitcoin Alpha and trust each other,

Table 2

Properties of the example real-world graphs used in this paper. We show the number of nodes and edges (n, m), if the graph is directed, density $d(G)$, diameter $\Delta(G)$, average clustering coefficient $C(G)$, average shortest path length $a(G)$, small-world coefficient $\sigma(G)$ and s-metric $s(G)$.

	E_1	E_2	E_3
n	57	475	3,783
m	637	10,222	24,186
directed	–	–	✓
$d(G)$	0.23	0.09	0.001
$\Delta(G)$	4	4	–
$C(G)$	0.26	0.30	0.16
$a(G)$	1.98	2.06	–
$\sigma(G)$	0.53	–	–
$s(G)$	–	–	–
$\omega(G)$	0.51	–	–

which is a directed graph. The weights were removed while retaining the directionality of the edges. This information is based on the works of Kumar, Hooi, Makhija, Kumar, Faloutsos, and Subrahmanian (2018), Kumar, Spezzano, Subrahmanian, and Faloutsos (2016).

We selected the first two networks because they exemplify typical social networks with supplemental data, making them suitable for also considering them as knowledge graphs. We added the third graph to offer another viewpoint on directed graphs. In Table 2, we present a summary of the essential graph metrics. We employ the small-world coefficient $\sigma(G)$ to assess if the graphs are small-world networks (Humphries & Gurney, 2008) and the s-metric to determine if they are scale-free networks (Li, Alderson, Doyle, & Willinger, 2005).

Networks E_1 and E_2 have a different size, but similar topological features. However, their density is different.

3.2. Random graphs

We evaluate centrality measures on random graphs. First, we consider Scale-Free Networks with nodes (Jackson, 2010). Moreover, Bollobás et al. introduced a widely used graph model with three random parameters $\alpha + \beta + \gamma = 1$ (Bollobás, Borgs, Chayes, & Riordan, 2003). These values define probabilities and thus define attachment rules to add new vertices between either existing or new nodes. This model allows loops and multiple edges, where a loop denotes one edge where the endvertices are identical, and multiple edges denote a finite number of edges that share the same endvertices. Thus, we convert the random graphs to undirected graphs. For testing purpose, we scale the number of nodes n and use $\alpha = 0.41$, $\beta = 0.54$, and $\gamma = 0.05$ as suggested by Bollobás et al. (2003).

As the model generates explicitly directed networks, some concepts studied in this paper are restricted to undirected networks (e.g., closeness, betweenness). However, using these random graphs as undirected is a well-studied topic (Bollobás & Riordan, 2004). We chose this random graph model since it is generic and feasible for computer simulations for measuring and evaluation purposes (Bollobás et al., 2003; Kivelä et al., 2014). In a scale-free distribution, many nodes have a small degree and only few nodes have a very large degree: most nodes are hence lowly linked. Thus, these small-degree nodes lead to a few communities which are highly connected.

The second random graph uses a fixed degree distribution and is widely known as Watts–Strogatz or Newman–Watts–Strogatz small-world random graph (Newman, Barabási, & Watts, 2006; Watts & Strogatz, 1998). The algorithm to create such a graph takes a number of nodes, the number of nearest neighbors that form a ring topology and the probability for adding a new edge. A small-world graph contains only small average paths and thus has a small diameter. Some studies investigate the relation between scale-free and small-world networks, in particular the relationship between the average path length and local

clustering (Aarstad, Ness, & Haugland, 2013). In general, it is possible to generate scale-free networks with small-world attributes (Klemm & Eguiluz, 2002).

3.3. Error measures

Since we want to evaluate how graph structures and in particular measures change when additional information is stored or data is missing, we partition the graph into two parts that are represented by different colors: the original data is uncolored, while missing or extra data is represented by blue nodes. Thus, given a random graph $G = (V, E)$, a next step comprises a probability p_b for blue nodes and edges which leads to a graph G with blue nodes $B \subset V$ and edges $B' \subset E$. Let $\mathfrak{B} = (B, B')$ denote the blue subgraph which is the induced subgraph of B' where all nodes not in B are removed. We first compute the centrality measures for all non-blue nodes in $V \setminus B$ in the graph G . Subsequently, we compute those measures for all non-blue nodes in the reduced graph $G \setminus \mathfrak{B} := (V \setminus B, E \setminus B')$. See Fig. 1 for an illustration. Resultant, we obtain two vectors $c_1, c_2 \in \mathbb{R}^n$ where here, n is the number of nodes in $V \setminus B$. We denote c_i by $c_i = (c_i^1, c_i^2, c_i^3, \dots)$ to identify the elements in the vector. Note that the entries of c_i are ordered, i.e. $c_i^1 \geq c_i^2 \geq c_i^3, \dots$ and so on. They are thus vectors of node ranks according to a centrality measure.

However, we also need to introduce $c'_1, c'_2 \in \mathbb{R}^n$, which are the vectors containing the centrality measures sorted by the induced ordering of nodes in $V \setminus B = \{v_1, \dots, v_n\}$. Thus we denote

$$c'_i = (c'^1_i, c'^2_i, c'^3_i, \dots) = (c(v_1), c(v_2), c(v_3), \dots).$$

This approach combines the advantages of existing methods to manipulate graphs. First, it combines missing edges and nodes and thus it is not limited to pre-defined categories or samples of errors. Second, using different values for p_b , it can simulate different types of errors, from small exceptions to largely incomplete networks. Third, it spans a wide field of interpretations: We can either interpret $G \setminus \mathfrak{B}$ as the ‘original’ graph and G as its extension with errors or G as the ‘original’ graph and $G \setminus \mathfrak{B}$ as a detected graph omitting information. In addition, it covers the concept of multilayer networks, if we assume that both $G \setminus \mathfrak{B}$ and $\mathfrak{B}$ form one particular layer.

However, we are primarily interested in errors of centrality measures, which means, in turn, differences between c_1 and c_2 . To identify these errors, we follow the approach introduced in Dörpinghaus et al. (2022b) and define ϵ and ϵ_N :

$$e(i, c_1, c_2) = \begin{cases} 0 & c_1^i = c_2^i \\ 1 & c_1^i \neq c_2^i \end{cases}$$

That is, $e(i, c_1, c_2) = 1$ if the element on the i th position of c_1 differs from the element on the i th position in c_2 .

Let x be an element contained in every c_u , $u \in \{1, 2\}$. Then $p(x, c_u)$ denotes the predecessor of element x in c_u and $s(x, c_u)$ denotes the successor of x in c_u . If x is the first element in c_u , then $p(x, c_u) = \emptyset$. If x is the last element of c_u , then $s(x, c_u) = \emptyset$. With these definitions, we define ϵ_N :

$$e_N(x, c_1, c_2) = \begin{cases} 1 & \text{if } p(x, c_1) = \emptyset \text{ and } s(x, c_1) \neq s(x, c_2), \\ & \text{or } s(x, c_1) = \emptyset \text{ and } p(x, c_1) \neq p(x, c_2), \\ & \text{or } s(x, c_1) \neq s(x, c_2) \text{ and } p(x, c_1) \neq p(x, c_2), \\ 1/2 & \text{if } s(x, c_1) \neq s(x, c_2) \text{ and } p(x, c_1) = p(x, c_2), \\ & \text{or } s(x, c_1) = s(x, c_2) \text{ and } p(x, c_1) \neq p(x, c_2), \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we consider the predecessor of an element in c_1 and check if this element is still a predecessor of this element in c_2 , and analyze analogously the successor of an element.

With this, we define two error measures ϵ and ϵ_N :

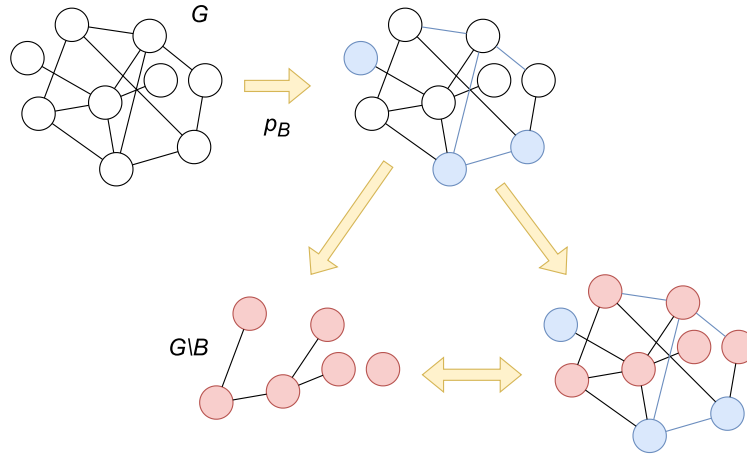


Fig. 1. Illustration of the construction of different graphs on the random graph G with p_B . In the first step, an arbitrary subset of G is marked as blue. In the next step, we can compare the nodes in B with their values in $G \setminus \mathfrak{B}$ (bottom left) and G (bottom right). We can either interpret $G \setminus \mathfrak{B}$ as the ‘original’ graph and G as its extension with errors or G as the ‘original’ graph and $G \setminus \mathfrak{B}$ as a detected graph omitting information.

$$\epsilon(c_1, c_2) = \frac{1}{n} \sum_{i=1}^n e(i, c_1, c_2)$$

$$\epsilon_N(c_1, c_2) = \frac{1}{n} \sum_{x \in c_1} e_N(x, c_1, c_2)$$

Example 1. Let $c_1 = (1, 2, 3, 4, 5)$, $c_2 = (5, 3, 2, 1, 4)$ and $c_3 = (1, 5, 2, 3, 4)$. If c_1 is the original ordering, we see that c_2 has a totally different order. In c_3 the entry 5 is moved, but the rest of the list is unchanged, although still 4 elements are on the wrong location. Hence, the number of misordered elements in c_1 compared to c_2 is 51. The number of moved elements is 5 and 1.

Here, $\epsilon(c_1, c_2) = 1$ and $\epsilon_N(c_1, c_2) = 1$. Moreover, $\epsilon(c_1, c_3) = \frac{4}{5}$ and $\epsilon_N(c_1, c_3) = \frac{1}{2}$.

3.4. Some aspects of critical graph structures

We will now consider two critical graph structures to take a closer look at their impact on the error measures on single nodes in these: Cliques and nodes v with $d(v) = 1$.

3.4.1. Cliques

Let $G = (V, E)$ be a graph on n nodes and $\mathfrak{B} = (B, B')$ denotes the blue subgraph on n nodes as introduced above. The nodes in $G \setminus \mathfrak{B}$ are denoted by $v_1, \dots, v_{n-n}$ while the nodes in $\mathfrak{B}$ are denoted by $v_{n-n+1}, \dots, v_n$. We further assume that $G \setminus \mathfrak{B}$ is still connected. Let $dc(G)$ be the vector containing the degree centrality measures for all nodes v in G in descending order, where – after the computation of $dc(v)$ for all $v_1, \dots, v_n \in V(G)$ – the values for all $v \in \mathfrak{B}$, that is, v_i with $i = n - n + 1, \dots, n$, are deleted. Hence,

$$dc(G) = (dc(v_1), dc(v_2), \dots, dc(v_{n-n}))$$

with $dc(v_j) \geq dc(v_{j+1})$ for all $j \in \{1, \dots, n - n\}$.

Let $bc(G)$ be the vector containing the betweenness centrality measures for all nodes in G in descending order, where – after the computation of $bc(v)$ for all $v_1, \dots, v_n \in V(G)$ – the values for all $v \in \mathfrak{B}$, that is, v_i with $i = n - n + 1, \dots, n$, are deleted. That is,

$$bc(G) = (bc(v_1), bc(v_2), \dots, bc(v_{n-n+1}))$$

with $bc(v_j) \geq bc(v_{j+1})$ for all $j \in \{1, \dots, n - n\}$. Let $p_{dc}(v)$ respectively $p_{bc}(v)$ be the position of node v in the vector $dc(G)$ respectively $bc(G)$. When it is clear from the context which vector is meant, we omit the index and simply write $p(v)$.

We may now prove some very basic observations on how a single blue node may influence the different error measures ϵ and ϵ_N , given that the blue node is part of a cluster in G . Here, with a cluster or a clique we denote a complete subgraph of G . All proofs are available in Appendix C.

Lemma 2. Let $G = (V, E)$ be a graph on n nodes and blue nodes $B \subset V$ and edges $B' \subset E$. Let $\mathfrak{B} = (B, B')$ denote the blue subgraph with $\mathfrak{B} = \{u\}$ where $G \setminus \mathfrak{B}$ is still connected. Let C_k be a clique in G with k nodes and let $u \in C_k$. Then

$$\epsilon(dc(G), dc(G \setminus \mathfrak{B})) \leq n - 1 - \min_{v \in N(u)} p_{dc}(v)$$

holds.

We can rely on the same basic observations for the error measure ϵ_N :

Lemma 3. Let $G = (V, E)$ be a graph on n nodes and blue nodes $B \subset V$ and edges $B' \subset E$. Let $\mathfrak{B} = (B, B')$ denote the blue subgraph with $\mathfrak{B} = \{u\}$ where $G \setminus \mathfrak{B}$ is still connected. Let C_k be a clique in G with k nodes and let $u \in C_k$. Then

$$\epsilon_N(dc(G), dc(G \setminus \mathfrak{B})) \leq k - 1$$

holds.

The stated lemma explains why this error increases for small-world networks: The node degree is high and a lot of local clusters exist.

Since betweenness centrality is also affected by the global structure of the graph, counting all shortest paths, the situation is slightly different.

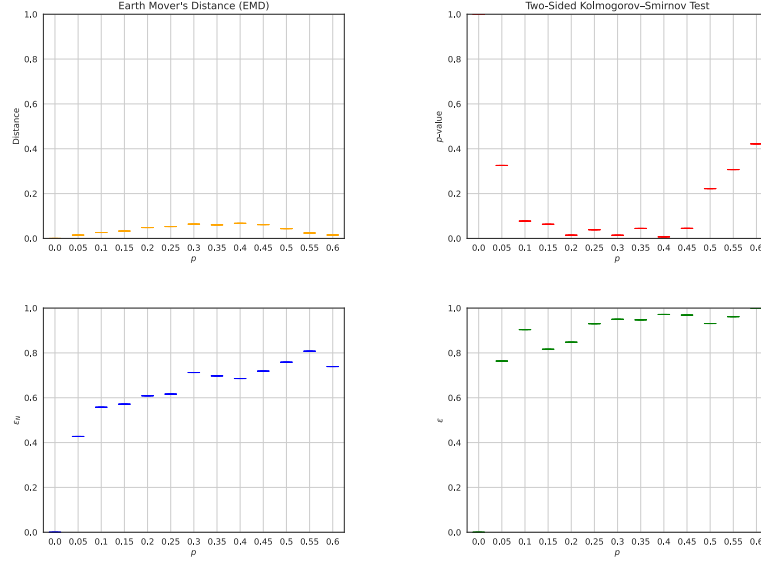
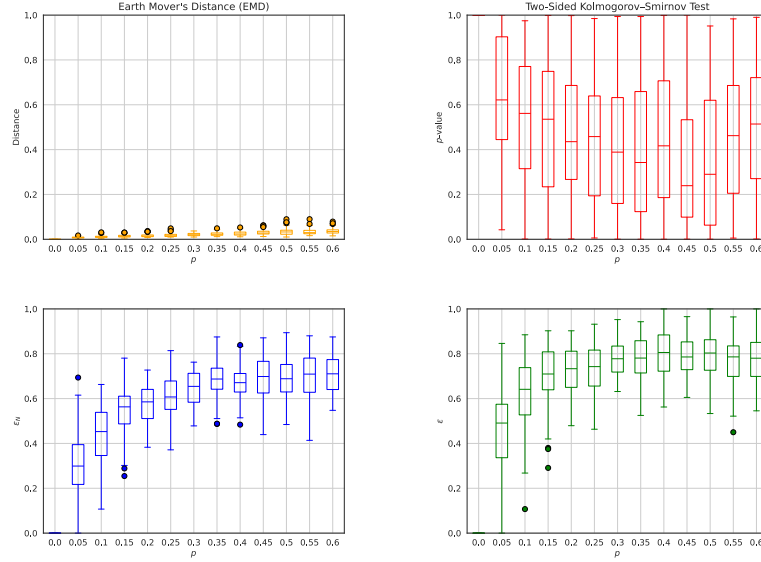
Lemma 4. Let $G = (V, E)$ be a graph on n nodes and blue nodes $B \subset V$ and edges $B' \subset E$. Let $\mathfrak{B} = (B, B')$ denote the blue subgraph with $\mathfrak{B} = \{u\}$ where $G \setminus \mathfrak{B}$ is still connected. Let C_k be a clique in G with k nodes and let $u \in C_k$. Then

$$\epsilon(bc(G), bc(G \setminus \mathfrak{B})) \leq \begin{cases} 0 & \text{if } d(u) = k - 1 \\ \sum_{w \neq y} P_u(w, y) & \text{otherwise.} \end{cases}$$

Here, we considered highly connected blue nodes within clusters and their impact on the error measures. Usually, external data may not only be added to such dense structures but also to single nodes. Thus, we will now discuss the impact of nodes with degree 1.

3.4.2. Nodes with degree 1

We now consider the special case in which the only existing blue node has degree 1. All proofs can be found in Appendix C.

Degree Centrality errors for Luke-Acts Network (E_1)(a) Choosing nodes with **largest degree centrality**.(b) Choosing nodes **randomly**.Fig. 2. Degree centrality errors for Luke-Acts network (E_1).

Lemma 5. Let $G = (V, E)$ be a graph on n nodes and blue nodes $B \subset V$ and edges $B' \subset E$. Let $\mathfrak{B} = (B, B')$ denote the blue subgraph with $\mathfrak{B} = \{u\}$ where $G \setminus \mathfrak{B}$ is still connected. Let further $N(u) = \{v\}$. Then

$$\epsilon(dc(G), dc(G \setminus \mathfrak{B})) \leq p_{dc}(v)$$

holds.

A similar observation can be made for ϵ_N :

Lemma 6. Let $G = (V, E)$ be a graph on n nodes and blue nodes $B \subset V$ and edges $B' \subset E$. Let $\mathfrak{B} = (B, B')$ denote the blue subgraph with $\mathfrak{B} = \{u\}$ where $G \setminus \mathfrak{B}$ is still connected. Let further $|N(u)| = 1$, that is, $d(u) = 1$. Then

$$\epsilon_N(dc(G), dc(G \setminus \mathfrak{B})) \leq 2$$

holds.

Since u lies only on the shortest paths between u and any other node in the graph, we can make the following observation:

Lemma 7. Let $G = (V, E)$ be a graph on n nodes and blue nodes $B \subset V$ and edges $B' \subset E$. Let $\mathfrak{B} = (B, B')$ denote the blue subgraph with $\mathfrak{B} = \{u\}$ where $G \setminus \mathfrak{B}$ is still connected. Let further $N(u) = \{v\}$ and let

$$t = \max \left\{ n - 1, \sum_{w \neq y} P_v(w, y) \right\}.$$

Then

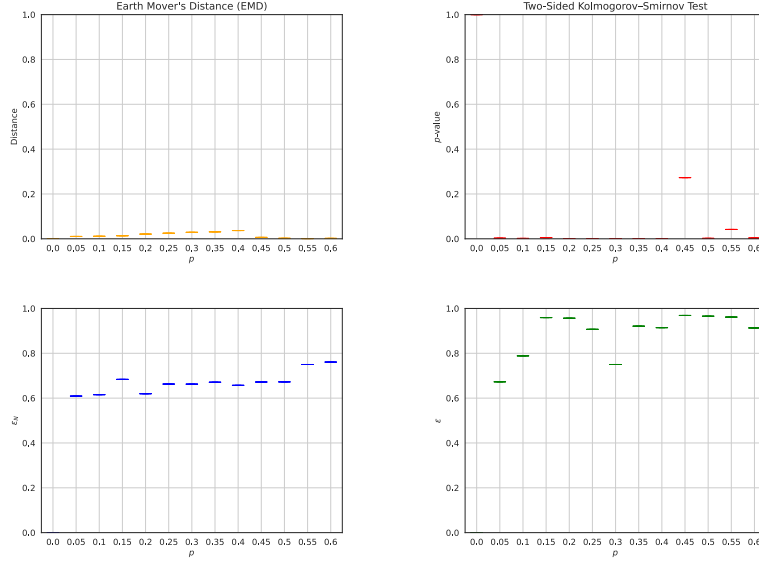
$$\epsilon(bc(G), bc(G \setminus \mathfrak{B})) \leq t$$

holds.

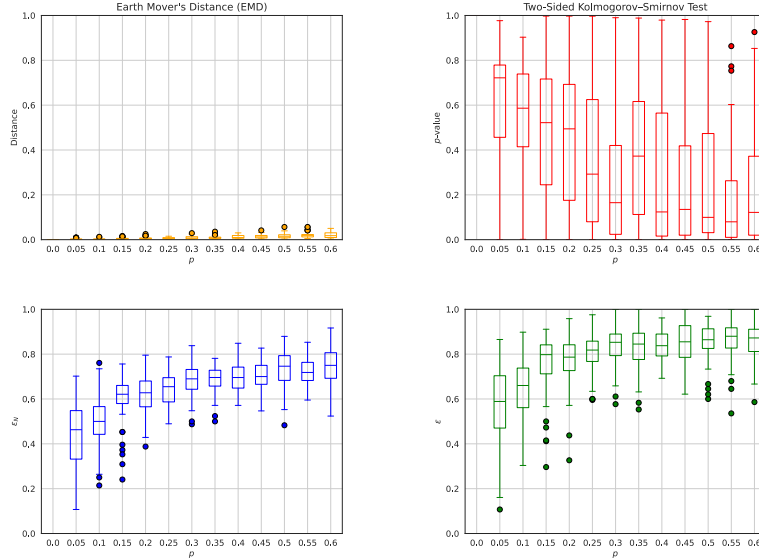
The evaluation of ϵ_N will – in general – decrease the worst-case scenario:

Lemma 8. Let $G = (V, E)$ be a graph on n nodes and blue nodes $B \subset V$ and edges $B' \subset E$. Let $\mathfrak{B} = (B, B')$ denote the blue subgraph with $\mathfrak{B} = \{u\}$ where $G \setminus \mathfrak{B}$ is still connected. Let further $N(u) = \{v\}$ and let

Betweenness Centrality errors for Luke-Acts Network (E_1)



(a) Choosing nodes with **largest betweenness centrality**.



(b) Choosing nodes Randomly.

Fig. 3. Betweenness centrality errors for Luke-Acts network (E_1).

$$z = \max \left\{ 2 \sum_{w \neq y} P_u(w, y), 2 \sum_{w \neq y} P_v(w, y) \right\}.$$

Then

$$\epsilon_N(bc(G), bc(G \setminus \mathfrak{B})) \leq z$$

holds.

These error estimations are not sharp. In addition, if the size of $\mathfrak{B}$ increases, it will be even more challenging to specify the error rates. But together with our experimental results, these estimations offer us a first impression of problematic graph structures having a great impact on ϵ and ϵ_N .

We could show that blue nodes in clusters have a great influence on both ϵ and ϵ_N while those nodes with a small neighborhood have a

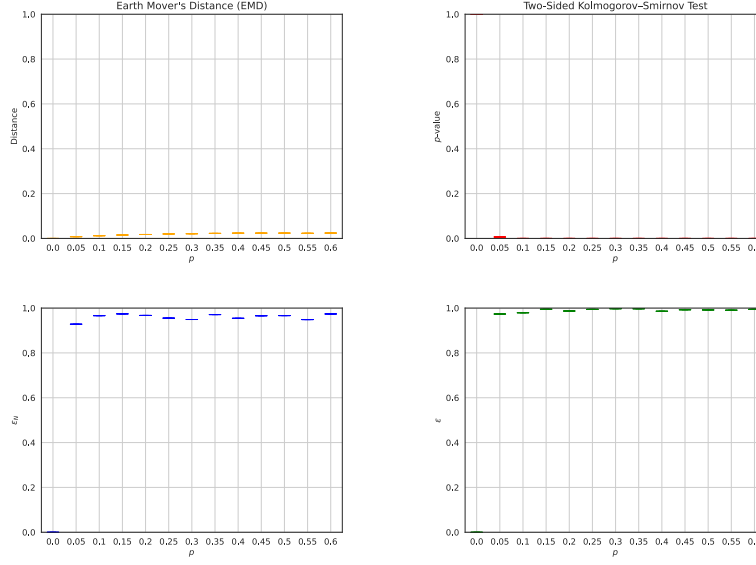
rather small influence on ϵ_N . This gives a first idea of why in general scale-free networks are more robust regarding ϵ_N . The degree centrality is only influenced by local structures but in general, the errors are higher while the betweenness centrality is in general more complex and the results here can only give some hints. Thus, we will now continue with experimental results and a detailed discussion.

4. Experimental results

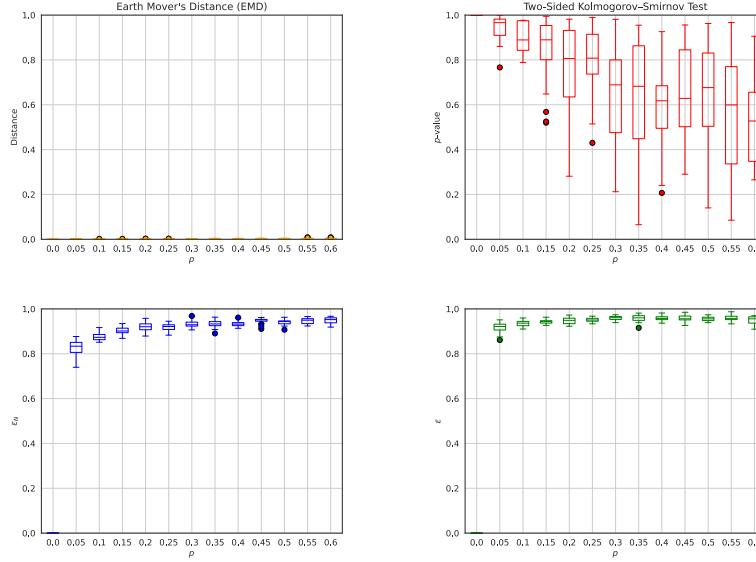
In this section, we will present the results of the approach introduced above. We will highlight and discuss interesting results and present a detailed discussion in the next section.

The centrality measures were evaluated with respect to the error measures ϵ and ϵ_N , Two-Sided Kolmogorov-Smirnov Test and Earth

Degree Centrality errors for Congress Network (E_2)



(a) Choosing nodes with **largest degree centrality**.



(b) Choosing nodes Randomly.

Fig. 4. Degree centrality errors for congress network (E_2).

Mover's Distance (EMD) for three real-world graphs, scale-free random graphs ($n = 200$, see Figs. 10, 11 right) and Newman–Watts–Strogatz small-world random graphs ($n = 200$, see Figs. 10, 11 left). Our analysis can either identify the error rate of missing or additional (either wrong or correct) data in the network. As discussed above, we considered the most popular centrality measures: Degree and Betweenness Centrality.

To evaluate the impact of missing or additional data, on centrality measures, we introduced statistical tests as robust tools. The Earth Mover's Distance (EMD) and Kolmogorov–Smirnov (K–S) test enable quantitative assessment of the discrepancies between centrality measures resulting from data errors. By applying these tests, we can effectively gauge the extent to which the network's structure and properties are affected by inaccuracies or omissions in the data, thereby providing valuable insights into the reliability and robustness of centrality measures in real-world network analyses. In we present details for selected experiments.

4.1. Error measures in real-world graphs

In our study of the real-world network, we implemented two distinct node removal strategies to assess their impact on the network's structural integrity. The first strategy involved randomly removing node fractions, while the second strategy focused on removing nodes based on their highest degree and betweenness centrality values, see Appendix D for details.

In the course of our analysis, we examined the impact of node removal strategies on sorting error ϵ and neighborhood error ϵ_N within the network. Initially, we focused on removing a fraction of nodes based on their largest degree centrality values, see Fig. 2(a). We observed that as the fraction of nodes removed increased, the sorting error consistently exhibited higher values compared to the neighborhood error. Furthermore, both errors demonstrated an increasing trend with the rise in the fraction of node removal.

Table 3
Network centrality and p-values (mean)

Network	Centrality	Elimination	Measure	Probability											
				0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6
Luke-Acts	Degree	Largest	ϵ	0.7636	0.9038	0.8163	0.8478	0.9302	0.95	0.9474	0.9714	0.9687	0.9310	0.9615	1.0
			ϵ_N	0.4273	0.5577	0.5714	0.6087	0.6163	0.7125	0.6974	0.6857	0.7187	0.7586	0.8077	0.7391
		Random	ϵ	0.4466	0.6232	0.7013	0.7276	0.7371	0.7821	0.7738	0.7988	0.7872	0.8015	0.7655	0.7753
			ϵ_N	0.2955	0.4433	0.5445	0.5767	0.6161	0.6476	0.6803	0.6707	0.6953	0.6927	0.6964	0.7064
	Betweenness	Largest	ϵ	0.6727	0.7885	0.9592	0.9565	0.907	0.75	0.9211	0.9143	0.9688	0.9655	0.9615	0.913
			ϵ_N	0.6091	0.6154	0.6837	0.6196	0.6628	0.6625	0.6711	0.6571	0.6719	0.6724	0.75	0.7609
		Random	ϵ	0.5672	0.643	0.7543	0.7716	0.8136	0.8418	0.8255	0.8385	0.8522	0.8551	0.8634	0.8609
			ϵ_N	0.448	0.5	0.6014	0.6198	0.6471	0.6814	0.6882	0.6968	0.704	0.7298	0.7254	0.7463
	Congress	Degree	ϵ	0.9735	0.979	0.995	0.9868	0.9944	0.997	0.9968	0.986	0.9924	0.9916	0.9907	0.9947
			ϵ_N	0.9281	0.9661	0.974	0.9671	0.9552	0.9489	0.9709	0.9544	0.9656	0.9664	0.9486	0.9737
		Random	ϵ	0.9171	0.9345	0.9432	0.948	0.9503	0.9593	0.958	0.9589	0.959	0.9562	0.9599	0.9515
			ϵ_N	0.825	0.8757	0.9036	0.9184	0.9177	0.9323	0.9328	0.9337	0.9465	0.9407	0.9467	0.9479
		Betweenness	ϵ	0.9712	0.9813	0.9851	0.9816	0.9832	0.994	0.9903	0.986	0.9924	0.9916	0.9813	0.9895
			ϵ_N	0.9746	0.9942	0.9765	0.9908	0.9818	0.9925	0.9935	0.986	0.9809	0.9832	0.986	0.9684
		Random	ϵ	0.9291	0.9476	0.9545	0.9594	0.9629	0.9627	0.9701	0.9654	0.9652	0.9666	0.9687	0.9721
			ϵ_N	0.9394	0.9554	0.9607	0.9681	0.9718	0.9678	0.9734	0.9699	0.9704	0.9709	0.9712	0.9658
Bitcoin	Degree	Largest	ϵ	0.9981	0.9991	0.9994	0.9993	0.9996	0.9981	0.9992	0.9991	0.9986	0.9974	0.9994	0.9967
			ϵ_N	0.6277	0.6775	0.7082	0.7019	0.6883	0.6657	0.6358	0.5989	0.5495	0.4886	0.4025	0.285
		Random	ϵ	0.9885	0.9903	0.9915	0.9922	0.9927	0.9928	0.993	0.993	0.9926	0.9924	0.9935	0.993
			ϵ_N	0.3101	0.4355	0.5164	0.5818	0.6206	0.6594	0.6831	0.7134	0.7323	0.7429	0.7646	0.7742
	Betweenness	Largest	ϵ	0.9989	0.9994	0.9997	0.9993	0.9996	0.9996	0.9996	0.9996	0.9875	0.9799	0.0	0.0
			ϵ_N	0.6329	0.6427	0.6354	0.6123	0.5851	0.5393	0.4829	0.4033	0.3054	0.1731	0.0	0.0
		Random	ϵ	0.9743	0.9835	0.9892	0.9896	0.9909	0.9925	0.9924	0.9926	0.9933	0.994	0.9934	0.9933
			ϵ_N	0.5379	0.5567	0.5732	0.5869	0.6026	0.6123	0.6265	0.6371	0.6436	0.6583	0.6737	0.6771
Small-world	Degree	Largest	ϵ	0.9576	0.973	0.9716	0.9774	0.9797	0.9779	0.9824	0.9835	0.985	0.9847	0.9856	0.9856
			ϵ_N	0.8636	0.9036	0.9093	0.9191	0.9246	0.9261	0.9271	0.9212	0.9313	0.9231	0.9194	0.9135
		Random	ϵ	0.9541	0.9632	0.9671	0.9703	0.9764	0.975	0.976	0.9809	0.9792	0.975	0.9763	0.9737
			ϵ_N	0.8356	0.89	0.9063	0.9204	0.9277	0.9294	0.9275	0.931	0.9301	0.9296	0.932	0.9288
	Betweenness	Largest	ϵ	0.9411	0.958	0.9682	0.9738	0.9736	0.9745	0.9772	0.9772	0.9785	0.9778	0.9796	0.9769
			ϵ_N	0.9538	0.9656	0.9708	0.9705	0.972	0.971	0.9714	0.9712	0.9686	0.9667	0.9639	0.9585
		Random	ϵ	0.9367	0.9523	0.9615	0.9619	0.9708	0.966	0.9663	0.9691	0.9702	0.974	0.9672	0.967
			ϵ_N	0.9461	0.9562	0.9649	0.9661	0.969	0.968	0.9675	0.9656	0.9661	0.9628	0.9595	0.9583
	Scale-free	Degree	ϵ	0.9704	0.9919	0.9935	0.9879	0.9554	0.8457	0.6569	0.3774	0.0305	0.0107	0.0	0.0
			ϵ_N	0.5853	0.5198	0.4829	0.4339	0.3574	0.2617	0.1847	0.086	0.0053	0.002	0.0	0.0
		Random	ϵ	0.7162	0.8836	0.9477	0.9527	0.9556	0.9562	0.9507	0.9441	0.9411	0.945	0.9429	0.9316
			ϵ_N	0.1328	0.2779	0.3354	0.4164	0.5186	0.5329	0.5586	0.5866	0.5926	0.5995	0.6301	0.628
		Betweenness	ϵ	0.9927	0.9924	0.9923	0.9685	0.929	0.5074	0.2622	0.0193	0.0	0.0	0.0	0.0
			ϵ_N	0.3882	0.35	0.3218	0.2416	0.1702	0.0617	0.0224	0.0024	0.0	0.0	0.0	0.0
		Random	ϵ	0.6564	0.8436	0.8862	0.9284	0.9408	0.9429	0.9565	0.9601	0.9595	0.9568	0.9608	0.9635
			ϵ_N	0.1609	0.2257	0.2735	0.2946	0.3219	0.3335	0.362	0.3823	0.3806	0.3911	0.416	0.403

Subsequently, we investigated the effects of removing nodes based on their highest betweenness centrality values, see Fig. 3(a). Similar to the previous strategy, we noted that the sorting error displayed higher values compared to the neighborhood error. Additionally, while the sorting error increased with the fraction of node removal, the neighborhood error remained relatively consistent even for higher fractions of removal.

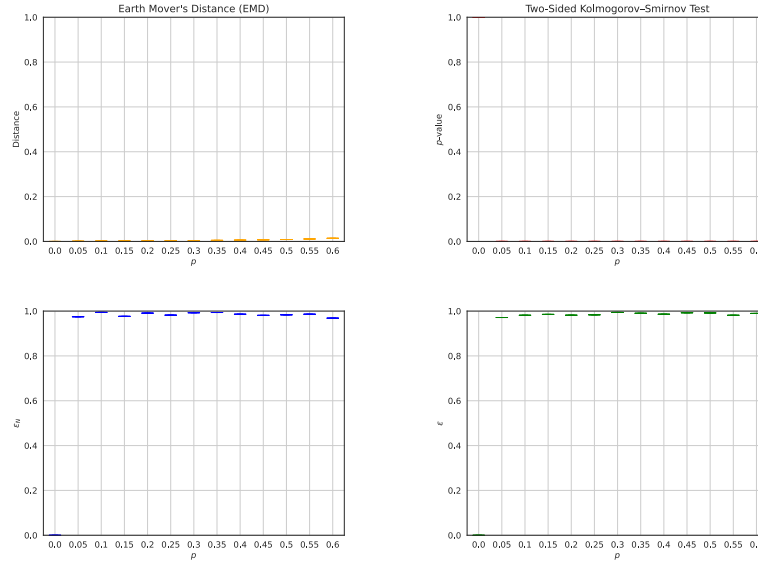
Lastly, we examined the outcomes of randomly removing nodes from the network. Once again, the sorting error exhibited higher values compared to the neighborhood error, and both errors displayed an increasing trend with the fraction of node removal.

In summary, our analysis revealed that regardless of the node removal strategy employed, the sorting error consistently displayed higher values compared to the neighborhood error. Additionally, both errors tended to increase with the fraction of node removal, highlighting the impact of node removal strategies on the network's structural integrity.

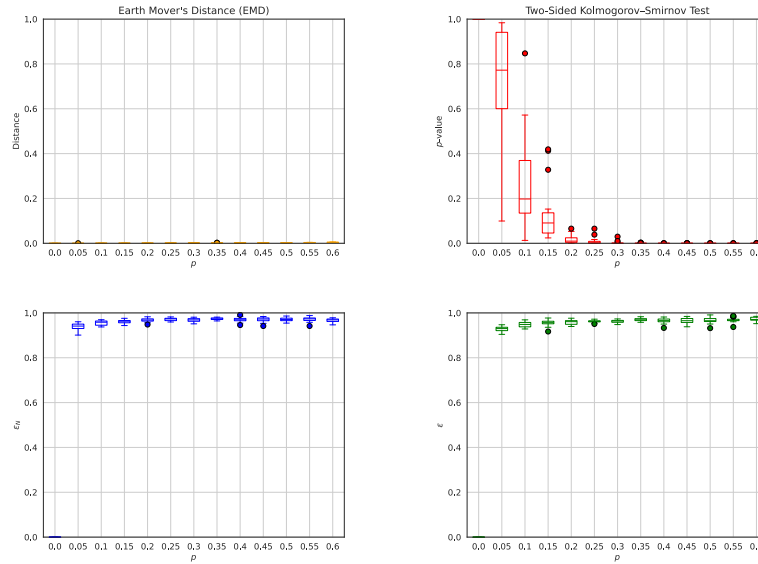
The analysis of the second network (E2) depicting the interactions within the US Congress on Twitter reveals noteworthy observation regarding sorting error ϵ and neighborhood error ϵ_N metrics.

The observations indicate that even a small fraction of node removal, particularly those with the highest centrality values, leads to

Betweenness Centrality errors for Congress Network (E_2)



(a) Choosing nodes with **largest betweenness centrality**.



(b) Choosing nodes Randomly.

Fig. 5. Betweenness centrality errors for congress network (E_2).

significantly higher sorting and neighborhood error rates; see Fig. 4(a) and Fig. 5(a). This discrepancy remains consistent even as the fraction of removed nodes increases, with both error rates approaching values closer to 1 and exhibiting relatively little variation across higher removal fractions.

Furthermore, the analysis suggests that the impact of node removal on the node order of members is substantial, regardless of whether nodes are removed randomly or based on centrality metrics. Even minor alterations in the network structure can cause notable shifts in the relative ordering of members within the network.

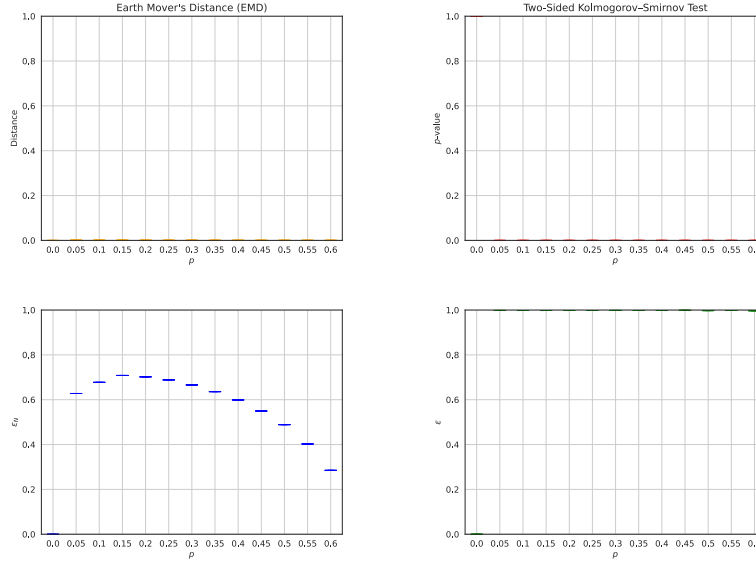
The analysis of the last real-world network (E3), which represents individuals trading Bitcoin on trading platforms, reveals notable patterns in sorting and neighborhood error metrics across various node removal strategies.

Throughout the analysis, it is observed that the sorting error consistently reaches its maximum value, close to 1, across all centrality removal strategies as well as random removal, see Fig. 6(a) and Fig. 7(a).

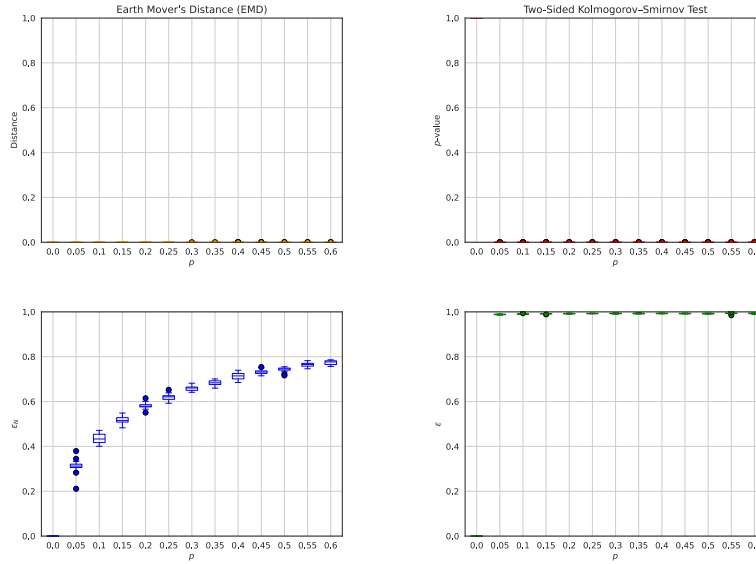
This indicates a substantial disruption in the relative ordering of nodes within the network regardless of the node removal approach.

However, the neighborhood error exhibits a different trend. For both nodes with largest degree centrality removal and largest betweenness centrality removal, the neighborhood error continues to decrease as the removal fraction increases, indicating a progressive improvement in the structural integrity of the network. Additionally, it is noted that the fraction of nodes removed based on largest betweenness centrality decreases more rapidly with higher removal fractions compared to removal based on largest degree centrality. Eventually, the order of nodes after removal based on largest betweenness centrality returns to its original state, indicating a restoration of the network's structural properties.

While EMD provides a quantitative measure of dissimilarity between distributions, interpreting the results in a meaningful way can be challenging. On the other hand P -value resulting from Kolmogorov-Smirnov test is easy to interpret.

Degree Centrality errors for Bitcoin Network (E_3)

(a) Choosing nodes with largest centrality.



(b) Choosing nodes Randomly.

Fig. 6. Degree centrality errors for Bitcoin Network (E_3).

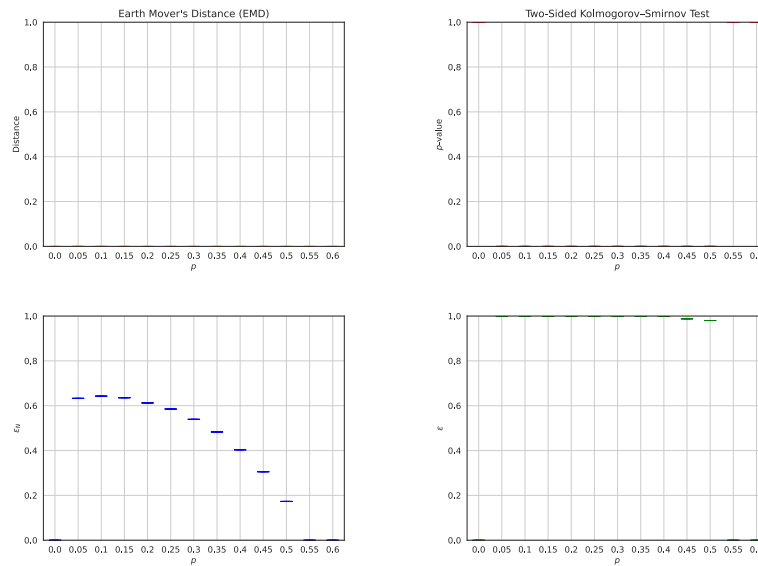
4.2. Error measures in random graphs

Centrality measures in scale-free graphs are not as severely influenced by missing data as the same measures in small-world random graphs. The obvious reason thereof is that small-world graphs are more connected and thus missing nodes easily alter a multitude of node centralities at once. We also note that the sorting error ϵ reacts more severely than the neighborhood error ϵ_N , which is also not surprising since the change in the ranking of a few nodes whirls around the sorting order more than the overall neighborhood relation.

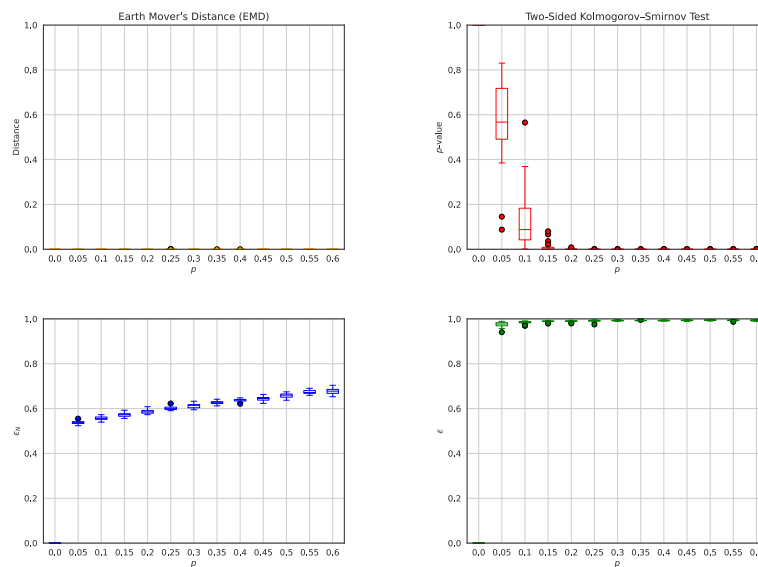
To get a better statistical understanding of the data, we compared 60 runs on different random graphs. The results are presented in boxplots, with Fig. 8 and Fig. 9 displaying the errors (ϵ and ϵ_N) observed in degree centrality and betweenness centrality for small-world networks, respectively. Similarly, Fig. 10 and Fig. 11 illustrate

the corresponding errors for scale-free networks. As these figures show, there is a significant difference between the different random graphs. This clearly shows that the stability of centrality measures also relies on the graph structure, a fact usually not considered in research: Scale-free graphs are usually much more stable, even for larger errors. In addition, the new error measure ϵ_N clearly shows that these networks are in particular stable preserving the order of actors — and small-world networks do not. With regards to the centrality measures, more local measure as degree centrality reacts less drastic to missing data as the more global measure betweenness centrality. This observation aligns with the intuitive understanding that localized measures primarily capture information from a node's immediate neighborhood, while global measures consider the entire network structure. However, scale-free networks show many outliers, especially for ϵ . This raises several questions, which we will discuss in the following.

Betweenness Centrality errors for Bitcoin Network (E_3)



(a) Choosing nodes with largest centrality.



(b) Choosing nodes Randomly.

Fig. 7. Betweenness Centrality errors for Bitcoin Network (E_3).

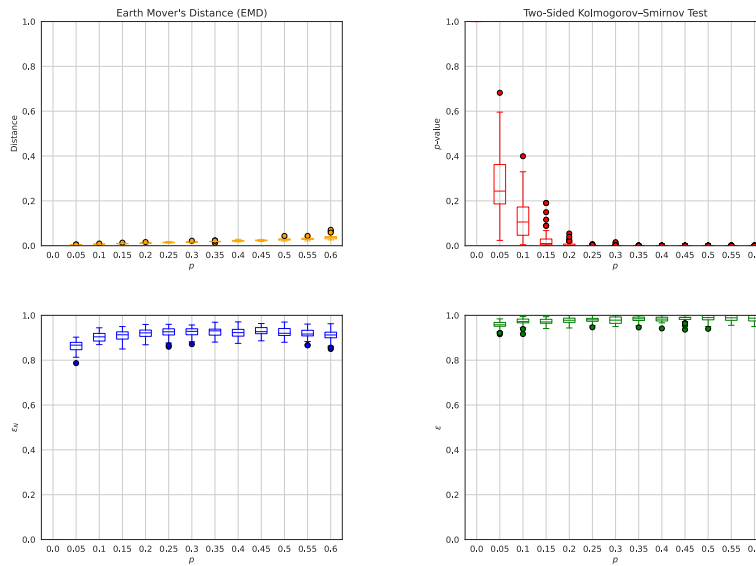
Our first observation is thus that scale-free networks are in general more robust against errors. However, the total number of errors increases fast. The error ϵ_N seems to be rather robust. Interestingly, in particular for betweenness centrality. Small-world networks are not very robust, albeit against small changes.

Previous researchers concluded that “from a global point of view, centrality is a sufficiently stable quantity to be used” as de Valeriola stated (de Valeriola, 2021). However, with our results, we could show that this fact is only true for scale-free networks. Thus, the robustness of centrality measures also relies on the network structure. If we assume social networks to behave like scale-free networks, we can support de Valeriola’s conclusion. However, there is an ongoing discussion about how real-world networks might look like and according to our knowledge, no further work has been carried out in this field, including

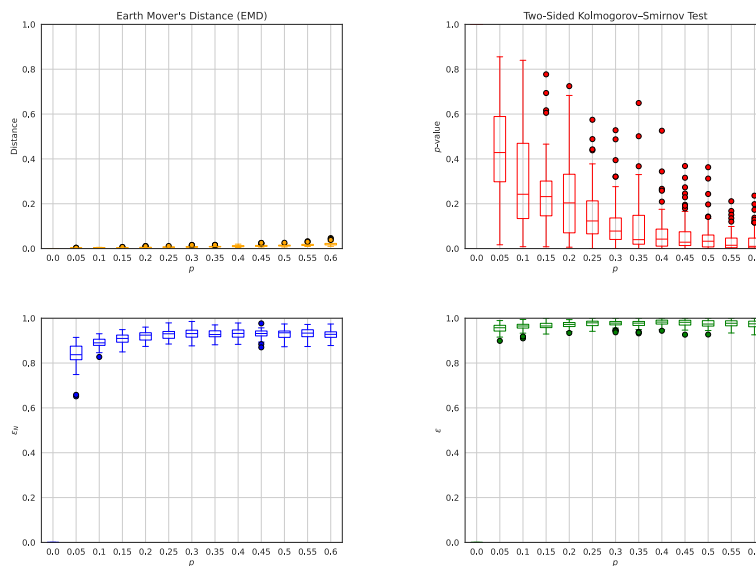
historical or religious networks. Furthermore, future work would be enhanced by an examination of the manner in which our findings contribute to the extant theories of network analysis and centrality. This could entail a correlation between the findings and theoretical models or frameworks in network science.

Interestingly, while some work was done on finding bound for centrality measures like betweenness centrality, see Gago, Hurajová, and Madaras (2012), Riondato and Kornaropoulos (2016), only very little research focused on the underlying graph structures that influence these measures. Very early research was carried out in Dörpinghaus et al. (2022b), Segarra and Ribeiro (2015). For betweenness centrality, a connection between the graph’s diameter and an upper bound for betweenness is known (Riordan et al., 2004). Since small-world and scale-free networks have a significantly different diameter, see Martel

Degree Centrality errors for Small-world networks



(a) Choosing nodes with largest centrality.



(b) Choosing nodes Randomly.

Fig. 8. Evaluation of Degree Centrality errors for Newman-Watts-Strogatz small-world random graphs with $n = 200$.

and Nguyen (2004), this might be a reason for their different behavior. However, it is still an open question which subgraph structures significantly change centrality values in general. More research needs to be done in this field.

However, the novel error measure ϵ_N , which was introduced by Dörpinghaus et al. (2022b), shows that scale-free networks are robust when only considering the general order of actors for small errors in the network. For small-world networks, the value for p_B needs to be significant small, at least $p_B \ll 0.01$. This directly leads to the question which error is important: Is it important to retrieve actors at the same position, or is it valid to preserve the order of actors?

For multi-layer networks, our analysis showed that they are in general not stable against new layers if we assume a significant size of a new layer. For example, adding locations as a new layer will lead to information which may well fill up 10% of the graph. However, then

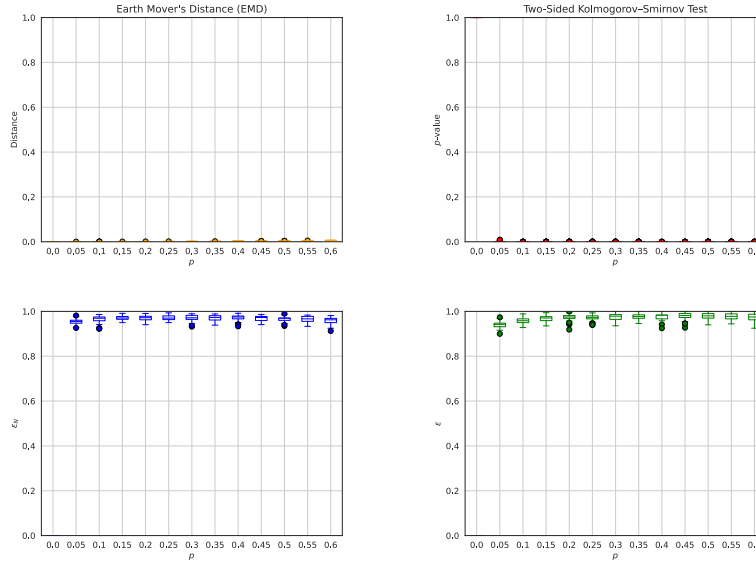
we can no longer speak of robustness. In particular, when adding more information layers to the graph, we need more discussion on this topic. In other words: Is SNA on multi-layer networks in general comparable and reproducible? Is there a consent in literature, which layers can be chosen? According to our best knowledge, no work has been carried out on this issue.

Summarizing, while de Valeriola is right to state that network analysis is always a compromise (de Valeriola, 2021), we may add that for historians, network analysis might be even more robust than expected, while in general more research needs to be done in this field.

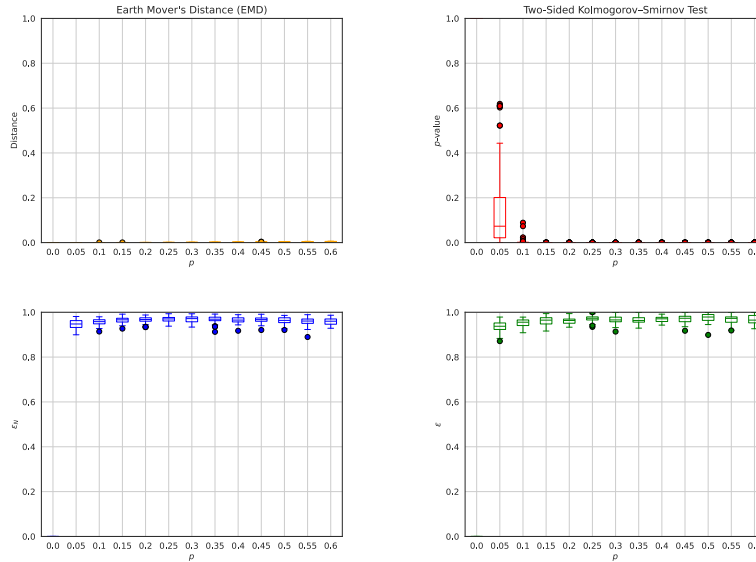
5. Conclusions and outlook

This paper introduces a novel, generic approach to analyze the robustness of centrality measures in networks for different node removal

Betweenness Centrality errors for Small-world networks



(a) Choosing nodes with largest centrality.



(b) Choosing nodes Randomly.

Fig. 9. Evaluation of Betweenness Centrality errors for Newman-Watts-Strogatz small-world random graphs with $n = 200$.

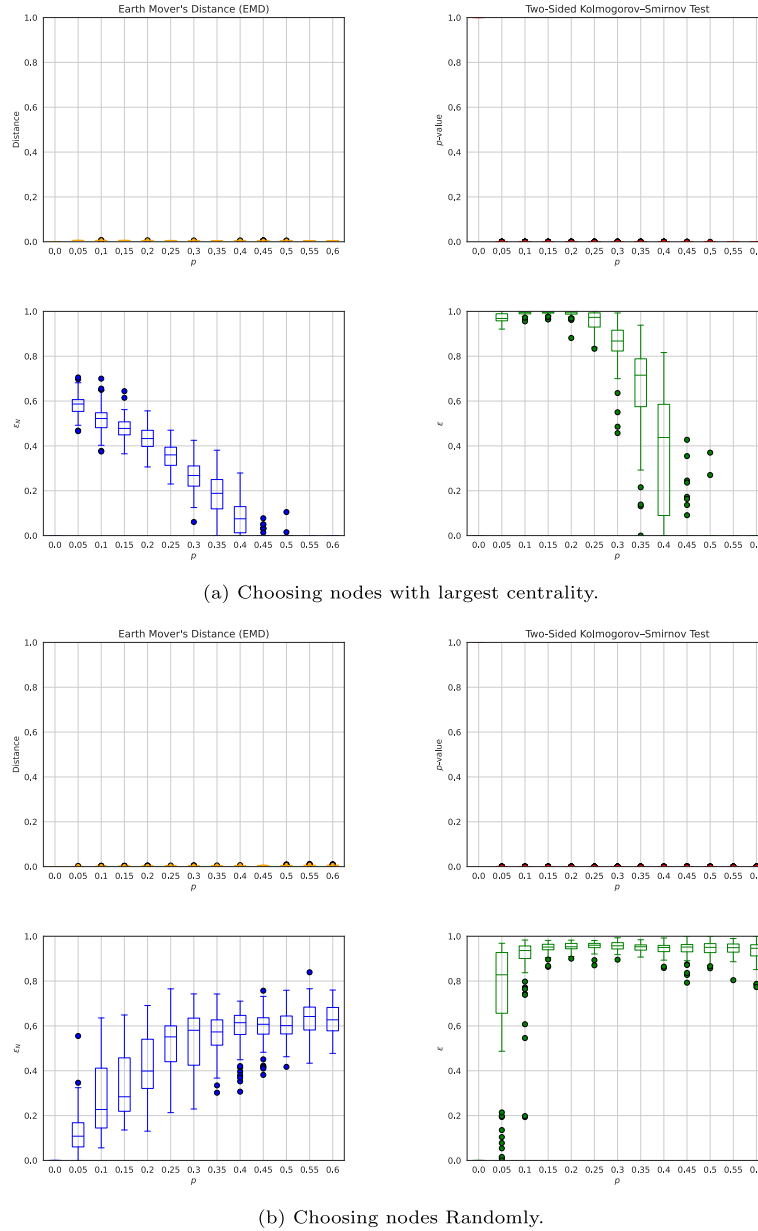
strategies, inspired by social network analysis. Our main contribution is the introduction of a novel error measure, to compute the robustness of a multi-layer social network against general errors, with the error measure ϵ and ϵ_N to account for misordered elements. However, we also presented an extensive discussion comparing four different error measures.

Our analysis of centrality measures aligns with previous work carried out. For example, de Valeriola claimed that his “conclusions are not always consistent with the literature” (de Valeriola, 2021), while he first studied graph theoretical artifacts as categories of errors that impacted errors in networks. However, till now, it is unclear how social networks in general and their degree distribution look like. As we can show, the topology of a graph has a serious impact on the robustness of centrality measures, and every claim on artifacts and errors should include a reference to this. In several disciplines, for example, historical sciences, the analysis is usually carried out on largely incomplete networks — historians, for example, never have

access to all sources, and some of them may also be lost. Thus, it is even more important to not only consider small artifacts but clearly analyze how large parts of missing data impact the network. This is especially the case in several application areas like epidemiology, telecommunications, or social media. In addition, since SNA is often carried out working with multi-layer networks, the impact of these layers also needs to be considered.

Our statistical analysis, utilizing the Earth Mover’s Distance (EMD) and Two-Sided Kolmogorov–Smirnov tests, suggests that the impact of node removal on the node order of members is substantial, regardless of whether nodes are removed randomly or based on centrality metrics. Even minor alterations in the network structure can cause notable shifts in the relative ordering of members within the network. However, the network structure influences this behavior, and further research needs to clarify this. In our analysis of random networks, we found that scale-free networks seem to be generally more robust to errors. This leads to a crucial question in the research process: it remains widely

Degree Centrality errors for scale-free networks

Fig. 10. Evaluation of Degree Centrality errors for scale-free random graphs with $n = 200$.

unknown or unknowable whether real-world networks, such as those in epidemiology or social media, follow the scale-free paradigm.

Comparing the novel error measures with Earth Mover's Distance (EMD) and the Two-Sided Kolmogorov–Smirnov Test, we obtained the following results: While EMD provides a quantitative measure of dissimilarity between distributions, interpreting the results in a meaningful way can be challenging. On the other hand, the P -value resulting from the Kolmogorov–Smirnov test is easy to interpret. The error measures ϵ and ϵ_N generally show fewer outliers and are more sensitive against changes, making them better accessible and interpretable.

In conclusion, when considered in comparison to the current state of the art, our results demonstrate the necessity for further research in this area. From one perspective, the presented generic approach represents a preliminary step towards subsequent comparisons and analysis. Conversely, the considerable diversity of networks and their topologies necessitates further investigation. The next logical step in

this research is to bridge the gap between graph theory and the application of network approaches. Although our work has demonstrated the potential for social sciences and humanities to benefit from social network analysis (SNA), several key questions remain unanswered. These include the specific types of graphs that can be used, the types of errors that may occur, and the methods for detecting such errors. Furthermore, we propose a comparative study that incorporates additional metrics and gives due consideration to computational efficiency. By employing a more expansive range of datasets, this approach would facilitate a more profound understanding of the efficacy of the proposed methodology. Our implementation is both operational and generic, yet it is still a work in progress. We hope that it will also highlight the importance of more interdisciplinary research in this field.

CRediT authorship contribution statement

Jens Dörpinghaus: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources,

Betweenness Centrality errors for scale-free networks

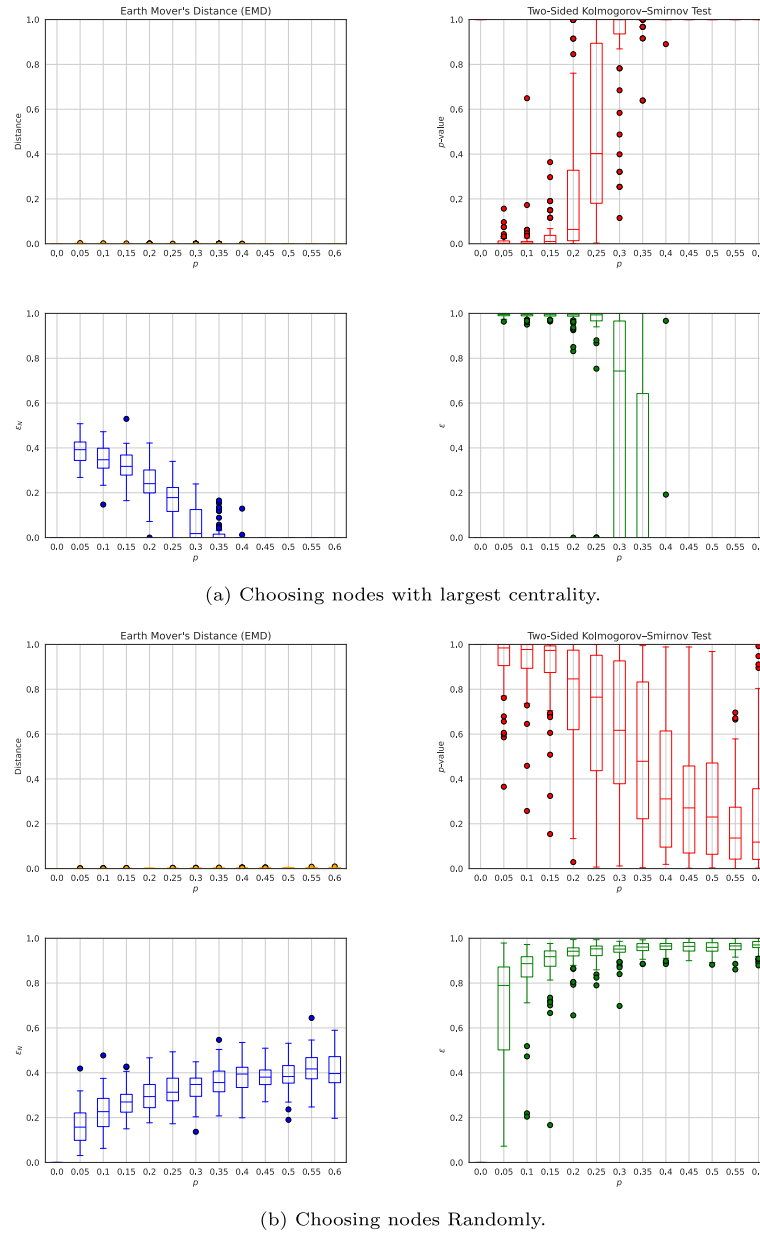


Fig. 11. Evaluation of Betweenness Centrality errors for scale-free random graphs with $n = 200$.

Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Vera Weil:** Writing – original draft, Resources, Methodology, Formal analysis, Conceptualization. **Robert Rockenfeller:** Writing – review & editing, Writing – original draft, Validation, Resources, Project administration, Methodology, Formal analysis. **Meerkumar Pravinbhai Mangoliya:** Writing – review & editing, Writing – original draft, Validation, Software, Methodology, Investigation, Formal analysis, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding

This research received no external funding. This article was funded by the Open Access Publication Fund of the Federal Institute for Vocational Education and Training (BIBB), Bonn.

Appendix A. Definition of centrality measures

In graph theory, the *neighborhood* of a vertex v is denoted by $N(v)$ and contains all nodes connected to v . If we are interested in the size of this neighborhood we calculate the node *degree* given by $\deg(v) = |N(v)|$.

The neighborhood thus gives information about the connectedness of an actor in the network. This can be useful to illustrate the direct influence of an actor within the complete network, especially for actors with a high node degree. But it is obvious that the amount of relations does not necessarily give a good idea on their quality or how we could

use these relations. While the node degree is often used as a measure to create random graphs, it is in general not a good measure in order to analyze particular actors in networks, see Jackson (2010).

Nevertheless, the *degree centrality* for a node $v \in V$ is given by

$$dc(v) = \frac{\deg(v)}{n-1}$$

The output value ranges between 0 and 1 and gives a reference to the direct connections. As discussed, it omits all indirect relations and in particular the node's position in the network. In contrast, *betweenness* analyzes critical connections between nodes and thus gives an indication of individuals that can change the flow of information in a network. This measure is based on paths in a network:

Much of the interest in networked relationships comes from the fact that individual nodes benefit (or suffer) from indirect relationships. Friends might provide access to favors from their friends, and information might spread through the links of a network, see Jackson (2010).

A *path* p in a graph $G = (V, E)$ is a set of vertices $v_1, \dots, v_t$, $t \in \mathbb{N}$, for example written as

$$p = [v_1, \dots, v_t],$$

where $(v_i, v_{i+1}) \in E$ for $i \in \{1, \dots, t-1\}$. The length $|p|$ of the path p is the total number of edges – not nodes. Thus $|p| = t-1$. The path p links the starting node v_1 and an ending node v_t . In a path, no crossings are allowed, thus $v_i \neq v_j$ for all $i, j \in \{1, \dots, t\}$, with $i \neq j$. If all properties of a path are met except that the beginning and the end vertex are the same – that is, $v_1 = v_t$ – we denote this set as a *circle*.

Betweenness centrality was first introduced by Freeman (1977) for symmetric relations – undirected graphs – and was extended to asymmetric relations – directed graphs – by White and Borgatti (1994). In the interpretation by social scientists, it considers other indirect connections by computing all possible shortest paths, see Schweizer (1996). Given a node v , importance is calculated by the number $P_v(k, j)$, that is, the number of all shortest paths in a network for all beginning and ending nodes $k, j \in V$ that pass through v divided by the number $P(k, j)$, the total number of shortest paths between k and j . After normalization the betweenness centrality is given by

$$bc(v) = \sum_{k \neq j, v \neq k, v \neq j} \frac{P_v(k, j)}{P(k, j)} \cdot \frac{2}{(n-1)(n-2)},$$

where n denotes the number of the vertices in the graph. This parameter allows an analysis of the critical links and how often a node lies on such a path. This centrality measure thus answers the questions whether a node can change the flow of information in a network or whether it is a bridge between other nodes, see Schweizer (1996).

A *Closure* describes closed groups, meaning clusters or communities, see Jackson (2010). The nodes within such a structure are nodes that can potentially establish trust within a cluster. The *Closeness* is determined by the number of shortest paths starting from a node. Thus, it indicates the potential possibility of a node to establish new connections. In contrast to betweenness, only selected paths and possibilities are considered here.

This leads to other occasionally used measures like *closeness centrality* and *harmonic closeness centrality*. Given a node $i \in V$ we can compute the average distance between the first and other nodes $j \in V$ with $\sum_{j \neq i} d(i, j)$ where $d(i, j)$ denotes the length of a shortest path between i and j . Then, according to Jackson (2010), we can compute closeness-centrality as follows:

$$cl(v) = \frac{n-1}{\sum_{u \in V} d(u, v)}$$

This is related to harmonic closeness-centrality:

$$hcl(v) = \sum_{u \in V, u \neq v} \frac{n-1}{d(u, v)}$$

While betweenness assumes network flows to be like packages flowing from a starting point to a destination, other measures consider multiple paths: For example, the so-called *eigenvector centrality* – introduced by Bonacich (1972) – measures the location of directly neighboring nodes in the network. For the eigenvector centrality, we “count walks, which assume that trajectories can not only be circuitous, but also revisit nodes and lines multiple times along the way.” Borgatti (2005). Thus, for the i -th node in the adjacency matrix A of a graph G , it is the i -th entry of x solving

$$Ax = \lambda x$$

with λ being the largest eigenvalue of A , see Girvan (2004). This measure not only classifies the direct possibility to influence neighbors, but also ranks the indirect possibility to influence the whole network. For a detailed mathematical background we refer to Jackson (2010).

Appendix B. Statistical tests

The *Kolmogorov-Smirnov (K-S) test* is a statistical tool employed to evaluate whether two cumulative distribution functions (CDF) differ significantly, see Arnold and Emerson (2011). It quantifies the maximum vertical difference between the CDF of the dataset of original degree distribution and the CDF of reduced degree distribution of the network after removal of nodes or edges.

Let $F_{1,n}$ be the cumulative distribution function of original network with a number of nodes n and $F_{2,m}$ be the cumulative distribution function of the network after node removal with a remaining number of nodes m , then the KS statistic, denoted as $D_{n,m}$, quantifies the maximum absolute difference between these functions, given by Arnold and Emerson (2011):

$$D_{n,m} = \sup_x |F_{1,n}(x) - F_{2,m}(x)|. \quad (1)$$

In practice, the null hypothesis is rejected at a specified significance level α if $D_{n,m}$ exceeds a critical value or in other words, if the condition in following equation (2) is satisfied, see Arnold and Emerson (2011).

$$D_{n,m} > c(\alpha) \cdot \sqrt{\frac{n \cdot m}{n+m}}. \quad (2)$$

The value of $c(\alpha)$ at specified significance level is tabulated in Wessel. A significant difference between the original and modified degree distributions is shown by a high K-S statistic suggesting that the removal approach has significantly affected the network's structure.

Earth Mover's Distance (EMD), also known as *Wasserstein distance*, is a metric used to measure the dissimilarity between two probability distributions, see Rubner, Tomasi, and Guibas (1998). It serves as a tool to measure the extent of structural modifications between two networks. This is achieved by quantifying the minimal amount of work, often conceptualized as the movement of elements, required to transform one distribution of centrality values into another.

When analyzing the impact of node removal on network centrality measures such as degree and betweenness centrality, EMD provides a quantitative measure of how much the centrality distribution of the network changes before and after the removal process. By comparing the centrality distributions before and after node removal, EMD captures the extent of structural alterations in the network topology.

The calculation of EMD stems from the solution of the classic transportation problem, see Jude et al. (2017) which involves finding an optimal transport plan that minimizes the total “work” required to transform one distribution into the other. Mathematically, let P and Q be two probability distributions, with m and n elements, respectively. Let c_{ij} represent the cost of transporting material from i in P to j in Q , and let f_{ij} denote the amount of material to be transported from i in P to j in Q . Then, the Earth Mover's Distance (EMD) is given by Rubner et al. (1998):

$$\text{EMD}(P, Q) = \frac{\sum_{i=1}^m \sum_{j=1}^n c_{ij} \cdot f_{ij}}{\sum_{i=1}^m \sum_{j=1}^n f_{ij}}$$

The optimal transport plan determines how much material should be moved from each bin in the source distribution to each bin in the target distribution, minimizing the overall transportation cost. Just like the transportation plan we need to decide on the minimum structural change that network underwent during node removal.

To conduct the statistical analysis on real-world and random networks, we utilize Python's Scipy library for statistical computations and networkx for network analysis. These tools enable robust evaluation and comparison of network properties and behaviors.

Appendix C. Proofs

Proof of Lemma 2:

Proof. Let $a_1 = dc(G)$ and $a_2 = dc(G \setminus \mathfrak{B})$. The only nodes which are affected by a decreasing degree centrality are those in the neighborhood $N(u)$ of the blue node u , since for $v \in N(u)$, only one node in the neighborhood of v is removed in $G \setminus \mathfrak{B}$ compared to G . Thus,

$$a_1^{p(v)} = a_2^{p(v)} - 1 \quad \forall v \in N(u)$$

holds. Observe that $\min_{v \in N(u)} p_{dc}(v)$ denotes the smallest position in $dc(G)$ of a node in $N(u)$ (that is, the highest ranked neighbor of u in $dc(G)$). All nodes in $dc(G)$, that are higher ranked are not affected by the deletion of u . Recall that $dc(G)$ only has $n - n = n - 1$ entries. Thus, at most $n - 1 - \min_{v \in N(u)} p_{dc}(v)$ nodes change their position in $dc(G \setminus \mathfrak{B})$ compared to $dc(G)$.

Proof of Lemma 3:

Proof. Let $a_1 = dc(G)$ and $a_2 = dc(G \setminus \mathfrak{B})$. Again, the only nodes which are affected by a decreasing degree centrality are those in the neighborhood of the blue node, that is the set $v \in N(u)$. Here, only one node in the neighborhood of these nodes is removed in $G \setminus \mathfrak{B}$. While the internal order of all nodes in $G \setminus \{C_k \setminus \{u\}\}$ does not change and the internal order of the $k - 1$ nodes in $C_k \setminus \{u\}$ remains untouched as well, at most the $k - 1$ nodes in $C_k \setminus \{u\}$ are shifted to a certain degree to the right, since their value in $dc(G \setminus \mathfrak{B})$ decreased compared to $dc(G)$. Every vertex in $C_k \setminus \{u\}$ hence contributes at most 1 to the sum computed in $\epsilon_N(dc(G), dc(G \setminus \mathfrak{B}))$, which leads to the upper bound $k - 1$.

Proof of Lemma 4:

Proof. Case 1 $d(u) = k - 1$: In this case, u only lies on shortest paths between u and any node in $G \setminus \{u\}$, since $N(u) = C_k \setminus \{u\}$. That is, every shortest path through C_k ignores u , since u is only connected to nodes within $C_k \setminus \{u\}$, see Fig. C.12.

Thus, the number of shortest paths in G that include u is $n - 1$, that is,

$$\sum_{w \neq y, w \neq u, y \neq u} P_u(w, y) = n - 1$$

holds. Hence, for $v \in G \setminus B$,

$$\sum_{w \neq y, w \neq u, y \neq u} P_v^{G \setminus \mathfrak{B}}(w, y) = \sum_{w \neq y, w \neq u, y \neq u} P_v^G(w, y) - 1$$

where P^G denotes a shortest path in the graph G . In other words: There exists a value $b \in \mathbb{R}$ such that for every $v \in G \setminus B$,

$$bc(G)^{p(v)} = bc(G \setminus \mathfrak{B})^{p(v)} - b.$$

Hence, the ordering of the values in $bc(G)$ compared to $bc(G \setminus \mathfrak{B})$ does not change and the considered difference is 0.

Case 2 $d(u) \geq k$: In this case, u lies on

$$s = \sum_{w \neq y, w \neq u, y \neq u} P_u(w, y)$$

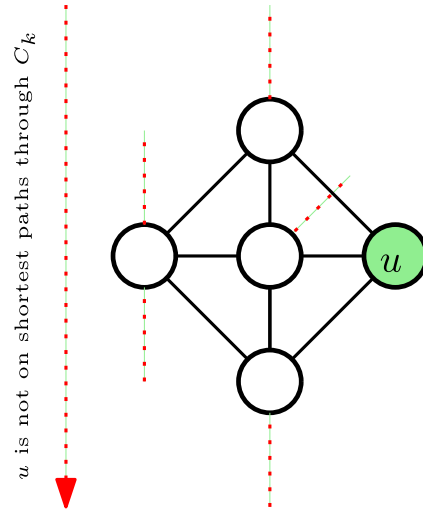


Fig. C.12. If $d(u) = k - 1$, u lies only on shortest paths between u and any other node in $G \setminus \{u\}$. Any shortest path through C_k ignores u , since u is only connected to nodes within C_k .

shortest paths within the graph. Thus, removing this node will affect all these paths and thus the ordering of at most s nodes will be changed. Hence $\epsilon(bc(G), bc(G \setminus \mathfrak{B})) \leq s$.

Proof of Lemma 5:

Proof. Since $d(u) = 1$, there is only one node v in $N(u)$. This node is on position $p_{dc}(v)$ in $dc(G)$ and it is the only node affected in $G \setminus \mathfrak{B}$. Thus, at most $p_{dc}(v)$ nodes are affected.

Proof of Lemma 6:

Proof. Let $a_1 = dc(G)$ and $a_2 = dc(G \setminus \mathfrak{B})$. If $d(u) = 1$, there is only one node v in $N(u)$. This node is on position $p_{dc}(v)$ in a_1 and it is the only node affected in $G \setminus \mathfrak{B}$. Thus, either it is at the same position in a_2 or on a different one which will affect the nodes on position $p_{dc}(u) - 1$ and $p_{dc}(u) + 1$ in a_1 and $p_{dc}(u) - 1$ and $p_{dc}(u) + 1$ in a_2 . Thus

$$\epsilon_N(dc(G), dc(G \setminus \mathfrak{B})) \leq 2$$

holds.

Proof of Lemma 7:

Proof. Let $a_1 = bc(G)$ and $a_2 = bc(G \setminus \mathfrak{B})$.

Since $d(u) = 1$, there is only one particular node $v \in N(u)$. Thus, all shortest paths containing u will include v . The highest impact of removing u will hence be on v . Moreover, $n - 1$ shortest paths that include u exist in $G \setminus \mathfrak{B}$. Thus, the first part of the maximum holds.

In general, $s = \sum_{w \neq y} P_u(w, y)$ shortest paths in G include v . Thus, removing u will also change the betweenness for s nodes and the second part of the maximum holds.

Proof of Lemma 8:

Proof. Let $a_1 = bc(G)$ and $a_2 = bc(G \setminus \mathfrak{B})$. Since $d(u) = 1$, there is only one particular node $v \in N(u)$. Thus, all shortest paths containing u will include v . Moreover, $n - 1$ such paths exist and the highest impact of removing u will be on v . But since we are interested in the ordering of nodes, the total number of reordered entries in a_2 may be just a factor. We can estimate this factor with the total number of shortest paths containing u which is $\sum_{w \neq y} P_u(w, y)$.

In general, again, $s = \sum_{w \neq y} P_v(w, y)$ shortest paths in G include v . Thus, removing u will also change the betweenness for s nodes and the second part of the maximum holds.

Appendix D. Algorithms

In our study, we implemented two distinct node removal strategies to assess their impact on the network's structural integrity. The first strategy involved randomly removing node fractions, while the second strategy focused on removing nodes based on their highest degree and betweenness centrality values. The time required for node removal is dependent on the data structure employed for graph implementation. However, it can be linear in the number of nodes.

remove_node_randomly(G)

```
1:  $i \leftarrow \text{random}(0, \text{len}(V))$ 
2:  $G.\text{remove\_node}(G.V[i])$ 
3: return  $G$ 
```

The removal of nodes based on their highest centrality measure necessitates the introduction of a further parameter, d , which is used to order the nodes according to this measure. While this does not affect the runtime of the function, it requires the recalculation of centrality measures after each step.

remove_highest_node(G, d)

```
1:  $i \leftarrow d[-1]$ 
2:  $G.\text{remove\_node}(G.V[i])$ 
3: return  $G$ 
```

Centrality measures for knowledge graphs are also quite complex. While several efficient algorithms have been proposed, see Grando et al. (2016), and some more specific problems are known to be $\mathcal{NP}$ -hard. For example Group Closeness Maximization (GCM), see Chen et al. (2016) or Maximum Betweenness Centrality, see Fink and Spoerhase (2011).

References

- Aarstad, J., Ness, H., & Haugland, S. A. (2013). In what ways are small-world and scale-free networks interrelated?. In *2013 IEEE international conference on industrial technology* (pp. 1483–1487). IEEE.
- Avenarius, C. B. (2010). Starke und schwache beziehungen. In C. Stegbauer, & R. Häußling (Eds.), *Handbuch Netzwerkforschung* (pp. 99–112). Wiesbaden: VS Verlag für Sozialwissenschaften.
- Berhan, L. M., Adams, A. L., McKether, W. L., & Kumar, R. (2019). Board 14: social networks analysis of African American engineering students at a pwi and an hbcu—a comparative study. In *2019 ASEE annual conference & exposition*.
- Bof, N., Baggio, G., & Zampieri, S. (2016). On the role of network centrality in the controllability of complex networks. *IEEE Transactions on Control of Network Systems*, 4, 643–653.
- Bollobás, B., Borgs, C., Chayes, J. T., & Riordan, O. (2003). Directed scale-free graphs. In *SODA* (pp. 132–139).
- Bollobás, B., & Riordan, O. (2004). Robustness and vulnerability of scale-free random graphs. *Internet Mathematics*, 1, 1–35.
- Borgatti, S. P., Carley, K. M., & Krackhardt, D. (2006). On the robustness of centrality measures under conditions of imperfect data. *Social Networks*, 28, 124–136.
- Carrington, P. J., Scott, J., & Wasserman, S. (2005). *Structural analyses in the social sciences: vol. 27, Models and methods in social network analysis*. Cambridge: University Press.
- Collar, A. (2013). *Religious networks in the roman empire*. Cambridge University Press.
- Costenbader, E., & Valente, T. W. (2003). The stability of centrality measures when networks are sampled. *Social Networks*, 25, 283–307.
- Craven, P., & Wellman, B. (1973). The network city. *Sociological Inquiry*, 43, 57–88.
- de Valeriola, S. (2021). Can historians trust centrality? *Journal of Historical Network Research*, 6.
- Dörpinghaus, J. (2021). Die soziale netzwerkanalyse: neue perspektiven für die auslegung biblischer texte?. *Biblisch Erneuerte Theologie*, 75–96.
- Dörpinghaus, J. (2022). Social network analysis of luke-acts. <http://dx.doi.org/10.5281/zenodo.7152121>, available at.
- Dörpinghaus, J., Binnewitt, J., Winnige, S., Hein, K., & Krüger, K. (2023). Towards a german labor market ontology: Challenges and applications. *Applied Ontology*, 1–23.
- Dörpinghaus, J., & Jacobs, M. (2019). Semantic knowledge graph embeddings for biomedical research: data integration using linked open data. In *SEMANTICS (Posters & Demos)*.
- Dörpinghaus, J., Klante, S., Christian, M., Meigen, C., & Düing, C. (2022a). From social networks to knowledge graphs: A plea for interdisciplinary approaches. *Social Sciences & Humanities Open*, 6, Article 100337.
- Dörpinghaus, J., & Stefan, A. (2019). Knowledge extraction and applications utilizing context data in knowledge graphs. In *2019 Federated conference on computer science and information systems* (pp. 265–272). IEEE.
- Dörpinghaus, J., & Stenschke, C. (2021). Ein kollaborativer workflow zur historischen netzwerkanalyse mit open source software. In *Proceedings of the 13th free and open source conference*.
- Dörpinghaus, J., Weil, V., Düing, C., & Sommer, M. W. (2022b). Centrality measures in multi-layer knowledge graphs. *Annals of Computer Science and Information Systems*, 32, 163–170.
- Düring, M. (2016). How reliable are centrality measures for data collected from fragmentary and heterogeneous historical sources? a case study. In *The connected past*. Oxford University Press.
- Easley, D., & Kleinberg, J. (2010). *Networks, crowds, and markets: reasoning about a highly connected world*. Cambridge: University Press.
- Ehrlinger, L., & Wölb, W. (2016). Towards a definition of knowledge graphs. In *SEMANTICS (Posters, Demos, SuCESS)*.
- Fensel, D., Şimşek, K., Huaman, E., Kärle, E., Panasiuk, O., Toma, I., et al. (2020). *Introduction: What is a knowledge graph?* (pp. 1–10). Cham: Springer International Publishing, http://dx.doi.org/10.1007/978-3-030-37439-6_1.
- Fink, C. G., Fullin, K., Gutierrez, G., Omodt, N., Zinnecker, S., Sprint, G., et al. (2023a). A centrality measure for quantifying spread on weighted, directed networks. *Physica A*.
- Fink, C. G., Omodt, N., Zinnecker, S., & Sprint, G. (2023b). A congressional twitter network dataset quantifying pairwise probability of influence. *Data in Brief*.
- Freeman, L. C. (1978). Centrality in social networks conceptual clarification. *Social Networks*, 1, 215–239.
- Gago, S., Hurajová, J., & Madaras, T. (2012). Notes on the betweenness centrality of a graph. *Mathematica Slovaca*, 62, 1–12.
- Ganter, A. (2015). Rezension: C. Rollinger: Amicitia sanctissima colenda. Freundschaft und soziale Netzwerke in der späten Republik. (Studien zur Alten Geschichte, Bd. 19). *Historische Zeitschrift*, 301, 182–184.
- Granovetter, M. S. (1973). The strength of weak ties. *American Journal of Sociology*, 78, 1360–1380.
- Hasheminezhad, R., & Brandes, U. (2023). Robustness of preferential-attachment graphs. *Applied Network Science*, 8, 1–19.
- Humphries, M. D., & Gurney, K. (2008). Network ‘small-world-ness’: a quantitative method for determining canonical network equivalence. *PLoS One*, 3, Article e0002051.
- Jackson, M. O. (2010). *Social and economic networks*. Princeton: Princeton University Press, <http://dx.doi.org/10.1515/9781400833993>.
- Kardos, O., London, A., & Vinkó, T. (2020). Stability of network centrality measures: a numerical study. *Social Network Analysis and Mining*, 10, 1–17.
- Kivelä, M., Arenas, A., Barthélemy, M., Gleeson, J. P., Moreno, Y., & Porter, M. A. (2014). Multilayer networks. *Journal of Complex Networks*, 2, 203–271.
- Klau, G. W., & Weiskircher, R. (2005). Robustness and resilience. In *Network analysis: Methodological foundations* (pp. 417–437). Springer.
- Klemm, K., & Eguiluz, V. M. (2002). Growing scale-free networks with small-world behavior. *Physical Review E*, 65, Article 057102.
- Kumar, S., Hooi, B., Makhija, D., Kumar, M., Faloutsos, C., & Subrahmanian, V. (2018). Rev2: fraudulent user prediction in rating platforms. In *Proceedings of the eleventh ACM international conference on web search and data mining* (pp. 333–341). ACM.
- Kumar, S., Spezzano, F., Subrahmanian, V., & Faloutsos, C. (2016). Edge weight prediction in weighted signed networks. In *Data mining (ICDM), 2016 IEEE 16th international conference on* (pp. 221–230). IEEE.
- Landherr, A., Friedl, B., & Heidemann, J. (2010). A critical review of centrality measures in social networks. *Business & Information Systems Engineering*, 2, 371–385.
- Leidwanger, J., Knappett, C., Arnaud, P., Arthur, P., Blake, E., Broodbank, C., et al. (2014). A manifesto for the study of ancient mediterranean maritime networks. *Antiquity*, 88.
- Li, L., Alderson, D., Doyle, J. C., & Willinger, W. (2005). Towards a theory of scale-free graphs: Definition, properties, and implications. *Internet Mathematics*, 2, 431–523.
- Ma, X., Wu, J., Xue, S., Yang, J., Zhou, C., Sheng, Q. Z., et al. (2021). A comprehensive survey on graph anomaly detection with deep learning. *IEEE Transactions on Knowledge and Data Engineering*.
- Martel, C., & Nguyen, V. (2004). Analyzing kleinberg’s (and other) small-world models. In *Proceedings of the twenty-third annual ACM symposium on Principles of distributed computing* (pp. 179–188).
- Martin, C., & Niemeyer, P. (2019). Influence of measurement errors on networks: Estimating the robustness of centrality measures. *Networking Science*, 7, 180–195.

- Martin, C., & Niemeier, P. (2020). The role of network size for the robustness of centrality measures. In *Complex networks and their applications VIII: Volume 1 Proceedings of the eighth international conference on complex networks and their applications complex networks 2019 8* (pp. 40–51). Springer.
- Newman, M. E., Barabási, A. L. E., & Watts, D. J. (2006). *The structure and dynamics of networks*. Princeton: University Press.
- Reitmayer, M., & Marx, C. (2010). Netzwerkansätze in der Geschichtswissenschaft. In C. Stegbauer, & R. Häußling (Eds.), *Handbuch Netzwerkforschung* (pp. 869–880). Wiesbaden: VS Verlag für Sozialwissenschaften.
- Riondato, M., & Kornaropoulos, E. M. (2016). Fast approximation of betweenness centrality through sampling. *Data Mining and Knowledge Discovery*, 30, 438–475.
- Riordan, O., et al. (2004). The diameter of a scale-free random graph. *Combinatorica*, 24, 5–34.
- Rodriguez, M. A., & Neubauer, P. (2010). Constructions from dots and lines. *Bulletin of the American Society for Information Science and Technology*, 36, 35–41.
- Rodriguez, M. A., & Neubauer, P. (2012). The graph traversal pattern. In *Graph data management: Techniques and applications* (pp. 29–46). IGI Global.
- Rollinger, C. (2014). Amicitia sanctissime colenda. In *Freundschaft und soziale Netzwerke in der Späten Republik*.
- Rollinger, C. (2020). Prolegomena. problems and perspectives of historical network research and ancient history. *Journal of Historical Network Research*, 4, 1–35.
- Rossetti, G., Citraro, S., & Milli, L. (2021). Conformity: A path-aware homophily measure for node-attributed networks. *IEEE Intelligent Systems*, 36, 25–34.
- Rubner, Y., Tomasi, C., & Guibas, L. (1998). A metric for distributions with applications to image databases. In *Sixth International Conference on Computer Vision (IEEE Cat. No.98CH36271)* (pp. 59–66). <http://dx.doi.org/10.1109/ICCV.1998.710701>.
- Ruegg, C., & Lee, J. J. (2020). Epic social networks and eve's centrality in milton's paradise lost. *Digital Scholarship in the Humanities*, 35, 146–159.
- Schweizer, T. (1996). *Muster sozialer Ordnung: Netzwerkanalyse als Fundament der Sozialethnologie*. D. Reimer, Berlin.
- Segarra, S., & Ribeiro, A. (2015). Stability and continuity of centrality measures in weighted graphs. *IEEE Transactions on Signal Processing*, 64, 543–555.
- Smith, J. A., & Moody, J. (2013). Structural effects of network sampling coverage i: Nodes missing at random. *Social Networks*, 35, 652–668.
- Smith, J. A., Moody, J., & Morgan, J. H. (2017). Network sampling coverage ii: The effect of non-random missing data on network measurement. *Social Networks*, 48, 78–99.
- Smith, J. A., Morgan, J. H., & Moody, J. (2022). Network sampling coverage iii: Imputation of missing network data under different network and missing data conditions. *Social Networks*, 68, 148–178.
- Suárez, D., Díaz-Puente, J. M., & Bettoni, M. (2021). Risks identification and management related to rural innovation projects through social networks analysis: A case study in Spain. *Land*, 10(613).
- Tsugawa, S., & Ohsaki, H. (2015). Analysis of the robustness of degree centrality against random errors in graphs. In *Complex networks VI* (pp. 25–36). Springer.
- Watts, D. J., & Strogatz, S. H. (1998). Collective dynamics of 'small-world' networks. *Nature*, 393, 440–442.
- White, D. R., & Borgatti, S. P. (1994). Betweenness centrality measures for directed graphs. *Social networks*, 16, 335–346.
- Xu, R., & Frank, K. A. (2021). Sensitivity analysis for network observations with applications to inferences of social influence effects. *Networking Science*, 9, 73–98.
- Zemljič, B., & Hlebec, V. (2005). Reliability of measures of centrality and prominence. *Social Networks*, 27, 73–88.
- Zhitomirsky-Geffet, M., & Prebor, G. (2019). Sagebook: toward a cross-generational social network for the Jewish sages' prosopography. *Digital Scholarship in the Humanities*, 34, 676–695.
- Žnidaršič, A., Ferligoj, A., & Doreian, P. (2018). Stability of centrality measures in valued networks regarding different actor non-response treatments and macro-network structures. *Networking Science*, 6, 1–33.